AP® CALCULUS BC
2008 SCORING GUIDELINES (Form B)

Question 6

Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

(a)
$$\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$$
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$$
$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$$

$3 : \begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.

1 : answer with reason

(c)
$$\ln\left(1+x^2\right) = \int_0^x \frac{2t}{1+t^2}\,dt$$
$$= \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$$
$$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$$

$2 : \begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \end{cases}$

(d)
$$\ln\left(\frac{5}{4}\right) = \ln\left(1+\frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$$

Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$.

Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0,

$$\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3}\cdot\frac{1}{64} < \frac{1}{100}.$$

$3 : \begin{cases} 1 : \text{uses } x = \frac{1}{2} \\ 1 : \text{value of } A \\ 1 : \text{justification} \end{cases}$

Form B
BC 6
6A$_1$

Work for problem 6(a)

$f(x) = 2x \cdot \frac{1}{1+x^2}$

$= 2x\left(1 - x^2 + x^4 - x^6 + \cdots\right)$ $\quad \cancel{-1<x^2<1}$

$= 2x - 2x^3 + 2x^5 - 2x^7 + \cdots \cancel{+2x^{2n+1}}$ $\quad \cancel{-1<x<1}$

$+ (-1)^n 2x^{2n+1}$

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Work for problem 6(b)

$f(x) = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$

$\rho = \lim_{n\to\infty} \left| \frac{2x^{2n+3}}{2x^{2n+1}} \right| = \lim_{n\to\infty} |x^2| < 1$ For the series to converge

$-1 < x < 1$

when $x = 1$

series $\sum_{n=0}^{\infty} (-1)^n 2$ $\quad \lim_{n\to\infty} |(-1)^n 2| = \lim_{n\to\infty} 2 \neq 0$

series diverges by divergent test

the series does not converge to $f(1)$ at $x=1$

Continue problem 6 on page 15.

Work for problem 6(c)

$f(x) = \frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1}$

$\ln(1+x^2) = \int_0^x f(t)\,dt$

$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots \quad -1 < x < 1$

Work for problem 6(d)

$\ln(\frac{5}{4}) = \ln(1+x^2) \quad x = \frac{1}{2}$

For $\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots + \frac{(-1)^{n+1} x^{2n}}{n}$

It is a alternating series.

Also for $-1 < x < 1$

$|a_n| > |a_{n+1}|$

$\lim_{n\to\infty} |a_n| = 0 \quad$ for $-1 < x < 1$

$\therefore$ The alternating series converges by the alternating series test.

$|A - \ln(\frac{5}{4})| < \frac{1}{100}$ when $n = 2$ $\quad a_{n+1} = \frac{(\frac{1}{2})^6}{3} = \frac{1}{192} < \frac{1}{100}$

$\therefore A = (\frac{1}{2})^2 - \frac{(\frac{1}{2})^4}{2} = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}$ //

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Form B
BC 6
6B₁

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Work for problem 6(a)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n}$$

$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + 2x^9 + (-1)^n 2x(x^{2n}) + \ldots$$

Work for problem 6(b)

Integral test

Ratio Test

$$\rho = \frac{t_{n+1}}{t_n} = \frac{(-1)^{\cancel{n+1}}\,\cancel{2x}(x^{\cancel{2n+2}})}{\cancel{(-1)^n}\,\cancel{2x}(x^{\cancel{2n}})} = -1(x^2)$$

$\lim_{n\to\infty} |\rho| = \lim_{n\to\infty} x^2 = x^2$ when $x = 1$, $x^2 = 1$

yes because the function of $f(x)$ when integrated converges to $f(1)$ when $x=1$.

Continue problem 6 on page 15.

Work for problem 6(c)

$\ln(1+x^2) = \int \frac{2x}{1+x^2}dx = \int 2x - \int 2x^3 + \int 2x^5 - \int 2x^7 + \ldots$

$\ln(1+x^2) = x^2 - \frac{2x^4}{4} + \frac{2x^6}{6} - \frac{2x^8}{8} + \frac{2x^{10}}{10} + \ldots$

Work for problem 6(d)

$1+x^2 = \frac{5}{4}$ when $x = \frac{1}{2}$.

evaluate $\ln(1+x^2)$ when $x = \frac{1}{2}$

$= \frac{1}{4} - \frac{2}{4}\left(\frac{1}{2}\right)^4 + \frac{2}{6}\left(\frac{1}{2}\right)^6 - \frac{2}{8}\left(\frac{1}{2}\right)^8 + \ldots$

= value it approaches (lets call it F)

$\frac{1}{100} - F < A < \frac{1}{100} + F$

Because the value it approaches is the value of $\ln(5/4)$, then you can find A as an interval between $\frac{1}{100} - F < A < \frac{1}{100} + F$. Since $\frac{1}{100} - F$ & $\frac{1}{100} + F$ are decimals, the rational # A in that interval can then be determined.

FORM B
BC6
6C1

NO CALCULATOR ALLOWED

Work for problem 6(a)

$f(x) = \frac{1}{1+x}$ $\quad f(0) = 1$ $\quad 1 - x + \frac{2x^2}{2!} - \frac{6x^3}{3!} \ldots$

$f'(x) = -\frac{1}{(1+x)^2}$ $\quad f'(0) = -1$ $\quad 1 - x + x^2 - x^3 \ldots (-1)^n (x)^n$

$f''(x) = \frac{2}{(1+x)^3}$ $\quad f''(0) = 2$

$f'''(x) = -\frac{6}{(1+x)^4}$ $\quad f'''(0) = -6$

For $f(x) = \frac{2x}{1+x^2}$

$1 - x^2 + x^4 - x^6 \ldots (-1)^n (x)^{2n+1}$

$2x - 2x^3 + 2x^5 - 2x^7 \ldots (-1)^n (x)^{2n+1}(2)$

$\therefore \boxed{2x - 2x^3 + 2x^5 - 2x^7 \ldots (-1)^n 2x^{2n+1}}$

Work for problem 6(b)

$\sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$

at $x = 1$

$\sum (-1)^n \cdot 2$

This series will not converge to $f(1)$. This is an alternating series. For it to converge, its limit as it approaches ∞ must be zero. This is not the case. The values of the sequence will oscillate between -2 and 2.

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Work for problem 6(c)

$f'(x) = \frac{2x}{1+x^2}$ $\int f'(x) = \ln(1+x^2)$

$\int 2x - 2x^3 + 2x^5 - 2x^7 \ldots (-1)^n(2)(x^{2n+1})\, dx$

$= \boxed{x^2 - \frac{2x^4}{5} + \frac{2x^6}{7} - \frac{2x^8}{8} \ldots \frac{(-1)^n 2x^{2n+2}}{2n+2}}$

Work for problem 6(d)

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Question 6

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). The student presents correct work in parts (a) and (c). In part (b) the student appeals to two different tests for convergence but makes an incorrect conclusion that the series “converges to $f(1)$.” In part (d) the student correctly uses $x = \frac{1}{2}$ but does not find a rational number A, so the last 2 points were not earned.

Sample: 6C
Score: 4

The student earned 4 points: 3 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). The student presents correct work in parts (a) and (b). In part (c) the student antidifferentiates the first term from part (a) correctly but does not antidifferentiate the other terms correctly, so no points were earned.