Question 4

Let f be the function given by $f(x) = kx^2 - x^3$, where k is a positive constant. Let R be the region in the first quadrant bounded by the graph of f and the x-axis.

(a) Find all values of the constant k for which the area of R equals 2.

(b) For $k > 0$, write, but do not evaluate, an integral expression in terms of k for the volume of the solid generated when R is rotated about the x-axis.

(c) For $k > 0$, write, but do not evaluate, an expression in terms of k, involving one or more integrals, that gives the perimeter of R.

(a) For $x \ge 0$, $f(x) = x^2(k - x) \ge 0$ if $0 \le x \le k$ $\int_0^k \left(kx^2 - x^3\right) dx = \left(\frac{k}{3}x^3 - \frac{1}{4}x^4\right)\Big\|_{x=0}^{x=k} = \frac{k^4}{12}$ Area $= \frac{k^4}{12} = 2$; $k = \sqrt[4]{24}$	$4: \begin{cases} 1: \text{integral} \\ 1: \text{antiderivative} \\ 1: \text{value of integral} \\ 1: \text{answer} \end{cases}$
(b) Volume $= \pi \int_0^k \left(kx^2 - x^3\right)^2 dx$	$2: \begin{cases} 1: \text{integrand} \\ 1: \text{limits and constant} \end{cases}$
(c) Perimeter $= k + \int_0^k \sqrt{1 + \left(2kx - 3x^2\right)^2}\, dx$	$3: \begin{cases} 1: \int_0^k \sqrt{1 + (f'(x))^2}\, dx \\ 1: \text{uses } f'(x) = 2kx - 3x^2 \text{ in integrand} \\ 1: \text{answer} \end{cases}$

Form B
BC 4
4A₁

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

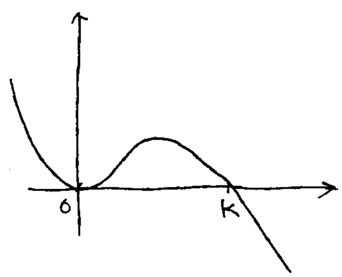
Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

$f(x) = kx^2 - x^3$

$= x^2(k-x) \quad (k>0)$



$R = \int_0^k f(x)\,dx$

$= \int_0^k (kx^2 - x^3)\,dx$

$= \frac{k}{3}x^3 - \frac{1}{4}x^4 \Big|_0^k$

$= \frac{1}{3}k^4 - \frac{1}{4}k^4 = \frac{1}{12}k^4 = 2$

$k^4 = 24$

$k = \sqrt[4]{24}$

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Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(b)

$\pi\int_0^k f(x)^2\,dx = \pi\int_0^k (kx^2-x^3)^2\,dx$

Work for problem 4(c)

$k+\int_0^k dl = k+\int_0^k \sqrt{1+(2kx-3x^2)^2}\,dx$

$f'(x) = 2kx-3x^2$

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Form B
BC 4
4B 1

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

Find the values of x where ~~though~~ $f(x) = 0$.

$f(x) = x^2(k-x)$.

Thus, $x = 0$ or k.

$R = \int_0^k f(x)\,dx$.

$= \frac{1}{3}kx^3 - \frac{1}{4}x^4 \Big|_0^k$

$= \frac{1}{3}k^4 - \frac{1}{4}k^4 - 0 = 2$.

$\frac{1}{12}k^4 = 2$

$k^4 = 24$

$k = \sqrt[4]{24}$ or $-\sqrt[4]{24}$

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Continue problem 4 on page 11.

4B2

Work for problem 4(b)

$\pi \int_0^k (kx^2 - x^3)\, dx$.

Work for problem 4(c)

The perimeter of R is the length of f(x) and the length of x-axis in the interval of (0, k).

That is: $k + \int_0^k \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= k + \int_0^k \sqrt{1+(2kx - 3x^2)}\, dx$

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Form B
BC 4
4 C_1

NO CALÇULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

$f(x) = kx^2 - x^3$

$\int f(x) = 2.$

$y = kx^2 - x^3$

x-intercept: $0 = kx^2 - x^3$

$0 = x^2(k - x)$

$x = k.$

$\int_0^k f(x) = 2.$

$\int_0^k kx^2 - x^3 = 2.$

$\left[\frac{1}{3}kx^3 - \frac{1}{4}x^4\right]_0^k = 2$

$(\frac{1}{3}k(k^3) - \frac{1}{4}k^4) - (0) = 2$

$\frac{1}{12}k^4 = 2.$

$k^4 = 24$

$k = \sqrt[4]{24}$

$= 2\sqrt[4]{6}$

$= \pm 2\sqrt{6}$

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Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

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Work for problem 4(b)

if $k>0$, $k = -2\sqrt{6}$ (rejected)(not applicable)

$V = \pi \int y^2 \, dx$

$= \pi \int (kx^2 - x^3)^2 \, dx$

$= \pi \int k^2x^4 - 2kx^5 + x^6 \, dx$

$= \pi [\frac{1}{5}k^2x^5 - \frac{2}{6}kx^6 + \frac{1}{7}x^6]$

$= \pi [\frac{1}{5}k^2x^5 - \frac{1}{3}kx^6 + \frac{1}{7}x^6]$

Work for problem 4(c)

Perimeter of R = k + y-intercept + length of curve.

y-intercept : $y = 0 - 0 = 0$

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Question 4

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 3 points in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the student presents correct work, but since $k > 0$ is specified in the problem, the answer point was not earned. In part (b) the student earned 1 point for the limits and constant, but because the square is not included on the integrand, the integrand point was not earned. In part (c) the student does not include the square on the integrand, and so the answer point was not earned.

Sample: 4C
Score: 4

The student earned 4 points: 3 points in part (a), 1 point in part (b), and no points in part (c). In part (a) the student presents correct work, but since $k > 0$ is specified in the problem, the answer point was not earned. In part (b) the student has a correct integrand but does not include any limits on the integral. In part (c) the student does not show enough calculus work to be eligible for any points.