Question 4

Let $f$ be the function given by $f(x) = kx^2 - x^3$, where $k$ is a positive constant. Let $R$ be the region in the first quadrant bounded by the graph of $f$ and the $x$-axis.

(a) Find all values of the constant $k$ for which the area of $R$ equals 2.

(b) For $k > 0$, write, but do not evaluate, an integral expression in terms of $k$ for the volume of the solid generated when $R$ is rotated about the $x$-axis.

(c) For $k > 0$, write, but do not evaluate, an expression in terms of $k$, involving one or more integrals, that gives the perimeter of $R$.

(a) For $x \geq 0$, $f(x) = x^2(k - x) \geq 0$ if $0 \leq x \leq k$

\[
\int_{0}^{k} (kx^2 - x^3) \, dx = \left[ \left( \frac{k}{3}x^3 - \frac{1}{4}x^4 \right) \right]_{x=0}^{x=k} = \frac{k^4}{12}
\]

Area $= \frac{k^4}{12} = 2$; $k = \sqrt[3]{24}$

(b) Volume $= \pi \int_{0}^{k} (kx^2 - x^3)^2 \, dx$

(c) Perimeter $= k + \int_{0}^{k} \sqrt{1 + (2kx - 3x^2)^2} \, dx$
Work for problem 4(a)

\[ f(x) = kx^2 - x^3 \]
\[ = x^2(k-x) \quad (k > 0) \]

\[ R = \int_0^k f(x) \, dx \]
\[ = \int_0^k (kx^2 - x^3) \, dx \]
\[ = \frac{k}{3} x^3 - \frac{1}{4} x^4 \bigg|_0^k \]
\[ = \frac{1}{3} k^4 - \frac{1}{4} k^4 = \frac{1}{12} k^4 = 2 \]
\[ k^4 = 24 \]
\[ k = \sqrt[4]{24} \]

Continue problem 4 on page 11.
Work for problem 4(b)

\[ \pi \int_{0}^{k} f(x)^2 \, dx = \pi \int_{0}^{k} (kx^2-x^2)^2 \, dx \]

Work for problem 4(c)

\[ k + \int_{0}^{k} \, dx = k + \int_{0}^{k} \sqrt{1 + (2kx-3x^2)^2} \, dx \]

\[ f'(x) = 2kx - 3x^2 \]

GO ON TO THE NEXT PAGE.
Work for problem 4(a)

Find the values of $x$ where $f(x) = 0$.

$f(x) = x^2(k-x)$.

Thus, $x = 0$ or $k$.

$$R = \int_0^k f(x) \, dx.$$  

$$= \frac{1}{3} k x^3 - \frac{1}{4} x^4 \bigg|_0^k$$  

$$= \frac{1}{3} k^4 - \frac{1}{4} k^4 - 0 = 2.$$  

$$\frac{1}{12} k^4 = 2$$  

$$k^4 = 24$$  

$$k = \sqrt[4]{24} \text{ or } -\sqrt[4]{24}$$  

Continue problem 4 on page 11.
Work for problem 4(b)

\[ a \int_{0}^{k} (kx^2 - x^3) \, dx \]

Work for problem 4(c)

The perimeter of \( R \) is the length of \( f(x) \) and the length of \( x \)-axis in the interval of \( (0, k) \).

That is:

\[ k + \int_{0}^{k} \sqrt{1 + (6kx - 3x^2)} \, dx \]

\[ = k + \int_{0}^{k} \sqrt{1 + (6kx - 3x^2)} \, dx \]
Work for problem 4(a)

\[ f(x) = kx^4 - x^3 \]

\[ \int f(x) = 2 \]

\[ y = kx^4 - x^3 \]

x-intercept: 0 = kx^3 - x^3

0 = x^3(k - 1)

x = k

\[ \int_0^k f(x) = 2 \]

\[ \int_0^k kx^4 - x^3 = 2 \]

\[ \left[ \frac{1}{5}kx^5 - \frac{1}{4}x^4 \right]_0^k = 2 \]

\[ \left( \frac{1}{5}k(k^5) - \frac{1}{4}k^4 \right) - 0 = 2 \]

\[ \frac{1}{5}k^4 = 2 \]

\[ k^4 = 10 \]

\[ k = \sqrt[4]{10} \]

\[ = 2\sqrt[4]{5} \]

\[ = \pm 2\sqrt[4]{5} \]

Continue problem 4 on page 11.
Work for problem 4(b)

\[ \text{if } k > 0, \quad k^2 - 25b \quad \text{rejected } 2(\text{not applic. } y) \]

\[ V = \pi \int y^2 \, dx \]
\[ = \pi \int (2x^3 - x^2)^2 \, dx \]
\[ = \pi \int 4x^6 - 4x^5 + x^4 \, dx \]
\[ = \pi \left[ \frac{4}{7}x^7 - \frac{2}{6}x^6 + \frac{1}{5}x^5 \right] \]
\[ = \pi \left[ \frac{4}{7}x^7 - \frac{1}{3}x^6 + \frac{1}{5}x^5 \right] \]

Work for problem 4(c)

Parameter of \( R = x + y \) with \( t = \) length of curve.

\( y = 0 \) is a point.

\( \int_{-\infty}^{\infty} \)

-11-
Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 3 points in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the student presents correct work, but since \( k > 0 \) is specified in the problem, the answer point was not earned. In part (b) the student earned 1 point for the limits and constant, but because the square is not included on the integrand, the integrand point was not earned. In part (c) the student does not include the square on the integrand, and so the answer point was not earned.

Sample: 4C
Score: 4

The student earned 4 points: 3 points in part (a), 1 point in part (b), and no points in part (c). In part (a) the student presents correct work, but since \( k > 0 \) is specified in the problem, the answer point was not earned. In part (b) the student has a correct integrand but does not include any limits on the integral. In part (c) the student does not show enough calculus work to be eligible for any points.