AP® CALCULUS AB
2008 SCORING GUIDELINES

Question 6

Let f be the function given by $f(x) = \dfrac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by $f'(x) = \dfrac{1 - \ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of f at $x = e^2$.

(b) Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

(c) The graph of the function f has exactly one point of inflection. Find the x-coordinate of this point.

(d) Find $\lim\limits_{x \to 0^+} f(x)$.

(a) $f\left(e^2\right) = \dfrac{\ln e^2}{e^2} = \dfrac{2}{e^2}$, $f'\left(e^2\right) = \dfrac{1 - \ln e^2}{\left(e^2\right)^2} = -\dfrac{1}{e^4}$ An equation for the tangent line is $y = \dfrac{2}{e^2} - \dfrac{1}{e^4}\left(x - e^2\right)$.	$2 : \begin{cases} 1 : f\left(e^2\right) \text{ and } f'\left(e^2\right) \\ 1 : \text{answer} \end{cases}$
(b) $f'(x) = 0$ when $x = e$. The function f has a relative maximum at $x = e$ because $f'(x)$ changes from positive to negative at $x = e$.	$3 : \begin{cases} 1 : x = e \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$
(c) $f''(x) = \dfrac{-\frac{1}{x}x^2 - (1 - \ln x)2x}{x^4} = \dfrac{-3 + 2\ln x}{x^3}$ for all $x > 0$ $f''(x) = 0$ when $-3 + 2\ln x = 0$ $x = e^{3/2}$ The graph of f has a point of inflection at $x = e^{3/2}$ because $f''(x)$ changes sign at $x = e^{3/2}$.	$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer} \end{cases}$
(d) $\lim\limits_{x \to 0^+} \dfrac{\ln x}{x} = -\infty$ or Does Not Exist	1 : answer

AB
6A1

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Work for problem 6(a)

$f(e^2) = \frac{\ln(e^2)}{e^2} = \frac{2}{e^2}$

$y - \frac{2}{e^2} = \left(\frac{1-\ln(e^2)}{e^4}\right)(x - e^2)$

$y - \frac{2}{e^2} = -\frac{1}{e^4}(x - e^2)$

Work for problem 6(b)

$f'(x) = 0 = \frac{1-\ln x}{x^2}$

$1 - \ln x = 0$ $\quad x^2 = 0$

$\ln x = 1$ $\quad \cancel{x = 0}$ not in domain

$x = e$

$f'(x)$ + −
e

Since $f'(x)$ changes from positive to negative @ $x = e$, this point is a relative maximum.

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NO CALCULATOR ALLOWED

6A2

Work for problem 6(c)

$$f''(x) = \frac{(x^2)(1-\ln x)' - (1-\ln x)(x^2)'}{x^4}$$

$$= \frac{-x - 2x(1-\ln x)}{x^4}$$

$$0 = \frac{-x - 2x + 2x\ln x}{x^4}$$

$$0 = -3x + 2x\ln x = x(2\ln x - 3)$$

$x = 0$ $\quad -3 + 2\ln x = 0$

$\ln x = \frac{3}{2}$

$\boxed{x = e^{3/2}}$

$f''(x)$ $-$ $+$ at $e^{3/2}$

There is an inflection pt @ $x = e^{3/2}$ because $f''(x)$ changes from $- \to +$ there.

Work for problem 6(d)

$$\lim_{x \to 0^+} \frac{\ln x}{x} = \frac{\ln(\text{really small pos \#})}{\text{really small pos \#}}$$

$$= \frac{\text{really Big neg \#}}{\text{really small pos \#}}$$

$=$ really, really big neg #

$\therefore \lim_{x \to 0^+} f(x) = -\infty$

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6B1

Work for problem 6(a)

$f(x) = \frac{\ln x}{x}$ $\quad f'(x) = \frac{1 - \ln x}{x^2}$

$f(e^2) = \frac{\ln e^2}{e^2} = \frac{2}{e^2}$ $\quad f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2} = \frac{1-2}{e^4} = -\frac{1}{e^4}$

$y = mx + b$

$\frac{2}{e^2} = -\frac{1}{e^{\cancel{4}}}(e^{\cancel{2}}) + b$

$\frac{2}{e^2} = -\frac{1}{e^2} + b$

$b = \frac{3}{e^2}$

$\boxed{y = -\frac{1}{e^4}x + \frac{3}{e^2}}$

Work for problem 6(b)

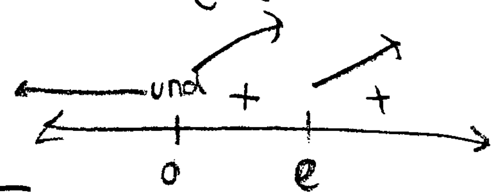
crit. point of $f \Rightarrow f'(x) = 0$

$\ln x^2$

$2 \ln x$

$\frac{1 - \ln x}{x^2} = 0$ $\quad$ und@ 0

$x = e$



criticall pt. $= \boxed{e}$ $\quad$ $\boxed{\text{neither}}$

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Continue problem 6 on page 15.

6B2

Work for problem 6(c)

$$f''(x) = \frac{\left(-\frac{1}{\cancel{x}}\right)\cancel{x^2} - (1-\ln x)2x}{x^4}$$

$$= \frac{-x - 2x(1-\ln x)}{x^4}$$

$$= \frac{-x - 2x + 2x\ln x}{x^4}$$

$$= \frac{-3 + 2\ln x}{x^3} = 0$$

$2\ln x = 3$

$\ln x = 3/2$

$\boxed{x = e^{3/2}}$

Work for problem 6(d)

L'hospitals

$$\lim_{x\to 0^+} \frac{\ln x}{x} = \frac{und}{und}$$

$$\frac{\frac{1}{x}}{1} = \frac{1}{x}$$

$\boxed{\infty}$

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6C₁

Work for problem 6(a)

$f(x)=\frac{\ln x}{x}$ $f'(x)=\frac{1-\ln x}{x^2}$

$f(e^2)=\frac{\ln e^2}{e^2}=\frac{2}{e^2}$

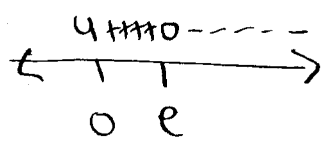
$(e^2, \frac{2}{e^2})$

$y-\frac{2}{e^2}=\left(\frac{1-\ln x}{x^2}\right)(x-e^2)$ $\boxed{y=\left(\frac{1-\ln x}{x}\right)-e^2\left(\frac{1-\ln x}{x^2}\right)+\frac{2}{e^2}}$

Work for problem 6(b)

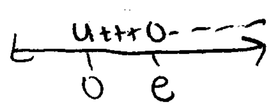
$f'(x)=(1-\ln x)(x^{-2})=0$

$\frac{(1-\ln x)}{x^2}=0$



$1-\ln x=0$

$1=\ln x$



$\boxed{x=e}$ this point is a relative maximum because the function is increasing $0<x<e$ and decreasing $e<x<\infty$.

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6C2

Work for problem 6(c)

$f'(x) = (1-\ln x)(x^{-2})$.

$f''(x) = (1-\ln x)(-2x^{-3}) + (x^{-2})\left(-\frac{1}{x}\right)$

$\frac{-2(1-\ln x)}{x^3} - \frac{1}{x^3}\left(\frac{-1}{x}\right) = \frac{-2(1-\ln x)-1}{x^3} = 0$

$-2 + 2\ln x - 1 = 0$

$2\ln x - 1 = 0$

$\ln x = \frac{1}{2}$

$\boxed{x = e^{\frac{1}{2}}}$

Work for problem 6(d)

$f(x) = \frac{\ln x}{x}$ L'Hopitals rule $\rightarrow \lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0}$ $\boxed{\text{undefined}}$

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Question 6

Overview

This problem presented students with a function f defined by $f(x) = \frac{\ln x}{x}$ for $x > 0,$ together with a formula for $f'(x).$ Part (a) asked for an equation of the line tangent to the graph of f at $x = e^2.$ In part (b) students needed to solve $f'(x) = 0$ and determine the character of this critical point from the supplied $f'(x).$ In part (c) students had to demonstrate skill with the quotient rule to obtain a formula for $f''(x)$ and solve $f''(x) = 0$ to find the x-coordinate of what was promised to be the only point of inflection for the graph of f. Part (d) tested students' knowledge of properties of $\ln x$ to determine the limit of $f(x)$ as $x \to 0^+.$

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 3 points in part (c), and no points in part (d). In part (a) the student identifies the value of the function and the derivative at $x = e^2$ and correctly writes a tangent line equation. In part (b) the student correctly identifies $x = e$ but classifies it as neither a minimum nor a maximum so only earned the first point. In part (c) the student gives the correct second derivative and correctly solves the equation. In part (d) the student's answer is not correct.

Sample: 6C
Score: 4

The student earned 4 points: no points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student does not identify the value of the derivative at $x = e^2$ and does not write an equation of the tangent line. In part (b) the student correctly identifies $x = e$ and classifies $x = e$ as a maximum but does not give a justification. The student gives only the definition of a maximum. In part (c) the student earned 2 points by correctly applying the product rule to find the second derivative. The student makes an arithmetic error in solving the equation and did not earn the third point. In part (d) the student has an acceptable answer but did not earn the point because of the reference to L'Hospital's Rule. This limit is not a candidate for L'Hospital's Rule.