Question 4

The functions f and g are given by $f(x) = \int_0^{3x} \sqrt{4+t^2}\, dt$ and $g(x) = f(\sin x)$.

(a) Find $f'(x)$ and $g'(x)$.

(b) Write an equation for the line tangent to the graph of $y = g(x)$ at $x = \pi$.

(c) Write, but do not evaluate, an integral expression that represents the maximum value of g on the interval $0 \le x \le \pi$. Justify your answer.

(a) $f'(x) = 3\sqrt{4+(3x)^2}$

$g'(x) = f'(\sin x) \cdot \cos x$

$= 3\sqrt{4+(3\sin x)^2} \cdot \cos x$

$4 : \begin{cases} 2 : f'(x) \\ 2 : g'(x) \end{cases}$

(b) $g(\pi) = 0,\ g'(\pi) = -6$

Tangent line: $y = -6(x-\pi)$

$2 : \begin{cases} 1 : g(\pi) \text{ or } g'(\pi) \\ 1 : \text{tangent line equation} \end{cases}$

(c) For $0 < x < \pi$, $g'(x) = 0$ only at $x = \dfrac{\pi}{2}$.

$g(0) = g(\pi) = 0$

$g\left(\dfrac{\pi}{2}\right) = \int_0^3 \sqrt{4+t^2}\, dt > 0$

The maximum value of g on $[0, \pi]$ is

$\int_0^3 \sqrt{4+t^2}\, dt.$

$3 : \begin{cases} 1 : \text{sets } g'(x) = 0 \\ 1 : \text{justifies maximum at } \dfrac{\pi}{2} \\ 1 : \text{integral expression for } g\left(\dfrac{\pi}{2}\right) \end{cases}$

Form B
AB 4
4A₁

NO CALCULATOR ALLOWED

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

$$f'(x) = \frac{d}{dx}\left[\int_0^{3x} \sqrt{4+t^2}\, dt\right]$$

$$= 3\sqrt{4+9x^2}$$

$$g'(x) = \frac{d}{dx}\left[\int_0^{3\sin x} \sqrt{4+t^2}\, dt\right]$$

$$= 3\cos x \sqrt{4+9\sin^2 x}$$

Do not write beyond this border.

Work for problem 4(b)

$$g'(\pi) = 3\cos\pi \sqrt{4+9\sin^2\pi}$$

$$= 3(-1)\sqrt{4+9(0)}$$

$$= -3(2)$$

$$= -6$$

$$g(\pi) = f(\sin\pi)$$

$$= f(0)$$

$$= 0$$

Equation of tangent line at $x = \pi$:

$$y = -6(x-\pi)$$

Do not write beyond this border.

Continue problem 4 on page 11.

Work for problem 4(c)

$g'(x) = 0$

$3\cos x \sqrt{4+9\sin^2 x} = 0$

$\Rightarrow 3\cos x = 0$

$x = \frac{\pi}{2}$

$x = \frac{\pi}{2}$ is a critical point

Since g is continuous and differentiable on the interval $[0, \pi]$, by Extreme Value Theorem, the global maximum can occur at the critical points or end points

$g\left(\frac{\pi}{2}\right) = f\left(\sin \frac{\pi}{2}\right)$

$= f(1)$

$= \int_0^3 \sqrt{4+t^2}\, dt > 0$

$\therefore$ The maximum value of g is $\int_0^3 \sqrt{4+t^2}\, dt$

$g(\pi) = f(\sin \pi)$

$= f(0)$

$= 0$

$g(0) = f(\sin 0)$

$= f(0)$

$= 0$

Do not write beyond this border.

Form B
AB4
4B1

NO CALÇULATOR ALLOWED

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Do not write beyond this border.

Work for problem 4(a)

$f'(x) = \sqrt{4 + (3x)^2} \cdot 3$

$= 3\sqrt{4 + 9x^2}$

$g'(x) = f'(\sin x) \cdot (\cos x)$

$= \cos x \cdot 3\sqrt{4 + 9\sin^2 x}$

$= 3\cos x \sqrt{4 + 9\sin^2 x}$

Work for problem 4(b)

$y = mx + b$

$m = g'(\pi) = 3\cos\pi\sqrt{4 + 9\sin^2\pi}$

$= -3\sqrt{4}$

$= -6$

$y = -6x + b$

$g(\pi) = f(\sin\pi) = f(0) = \int_0^0 \sqrt{4 - t^2}\,dt = 0$

$0 = -6\pi + b$

$b = 6\pi$

$\boxed{y = -6x + 6\pi}$

Do not write beyond this border.

Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

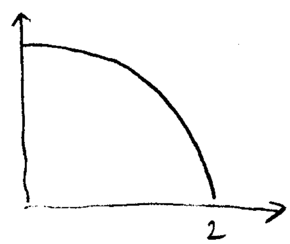
Work for problem 4(c)

$f(x) = \int_0^{3x} \sqrt{4+t^2}\, dt$

└ the formula for a circle with center (0,0) and radius of 2

Maximum value of g occurs when there is a maximum value of f.

f has its maximum value when $x = \frac{2}{3}$, because the answer will be the area of the quarter circle below.



This area is $= \int_0^2 \sqrt{4+t^2}\, dt$

∴ The maximum value is above

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Form B
AB4
4C₁

NO CALCULATOR ALLOWED

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

$f'(x) = \frac{d}{dx}\int_0^{3x} \sqrt{4+t^2}\,dt = 3\sqrt{4+(3x)^2}.$

$g'(x) = f'(\sin x) = 3\sqrt{4+(3\sin x)^2}$

Work for problem 4(b)

$g(\pi) = f(\sin\pi) = f(0) = 0.$ $(\pi, 0)$

$g'(\pi) = f'(\sin\pi) = 3\sqrt{4+(0)^2} = 6.$

$y - 0 = 6(x - \pi)$

$y = 6x - 6\pi$

Do not write beyond this border.

Do not write beyond this border.

Continue problem 4 on page 11.

4C2

NO CALCULATOR ALLOWED

Work for problem 4(c)

$$\int_0^{\pi} 3\sqrt{4+(3\sin x)^2}\;dx$$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

AP® CALCULUS AB
2008 SCORING COMMENTARY (Form B)

Question 4

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 4 points in part (a), 2 points in part (b), and no points in part (c). The student presents correct work in parts (a) and (b). In part (c) the student tries to argue from a geometric point of view, but the initial premise is incorrect, so no points were earned.

Sample: 4C
Score: 4

The student earned 4 points: 2 points in part (a), 2 points in part (b), and no points in part (c). In part (a) the student has a correct $f'(x)$ but makes a chain rule error in finding $g'(x)$ so earned just 2 of the 4 points. In part (b) the student finds $g(\pi)$ correctly and finds a value for $g'(\pi)$ based on the incorrect answer in part (a). The student combines these values to form a tangent line equation, earning both points in part (b). The student's work in part (c) did not earn any points.