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The Immortal L'Hospital

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Who Was This Man?

Each year the name of Guillaume-François-Antoine L'Hospital, Marquis de Sainte-Mesme, Comte d'Entremont, lives on in every first-year calculus classroom throughout the world. Sometimes students never hear of Gottfried Wilhelm Leibniz or Johann Bernoulli, but they all know the name L'Hospital. Who was this man who achieved such immortality in the calculus classroom? Where did he come from? Is his fame justified? What are his real contributions to mathematics?

Guillaume-François-Antoine L'Hospital. This image appears in the MacTutor History archive (http://www-history.mcs.st-and.ac.uk/history), a Web site of the School of Mathematics and Statistics at the University of St. Andrews in Scotland.

Guillaume-François de L'Hospital was born in Paris in 1661 and died there on February 2, 1704. Do not be confused about the spelling of his last name. He was also known as the Marquis de L'Hospital, and later the family spelled the name LHospital and still later L'Hôpital. (Consequently, this last spelling of his last name is commonly used -- as in "L'Hôpital's rule" -- despite being incorrect.) As expected of a man born into a wealthy, socially important family, he joined the army, rising to the rank of captain. However, his nearsightedness caused him to resign from the army. Early in his life he became a member of a scientific circle of amateur mathematicians led by Nicolas Malebranche (1638-1715) that included Louis Carré, Charles René Reyneau, and Pierre Varignon. We don't know much about his life prior to 1690. A portrait of him hangs in a room at the University of Padua. Because this room features portraits of renowned students in science and mathematics, historians surmise that L'Hospital studied at Padua for a time.

In any event, L'Hospital participated in a mathematics group in Paris at a very exciting time. Leibniz had published two short papers in the *Acta Eruditorum*, one in 1684 on the differential calculus and the second in 1686 on the integral calculus. Written in a very obscure manner, his papers were full of misprints and difficult to understand. Readers found the paper on the differential calculus to be somewhat understandable but the second paper on the integral calculus almost impossible to decipher. Leibniz left Paris for some time after the publication of his papers. It took two brilliant Swiss brothers from Basel living in Paris at that time to make sense of Leibniz's work -- Jakob (Jacques) Bernoulli, 1654-1705, and Johann (Jean) Bernoulli, 1667-1748. Johann gave a series of lectures on the differential calculus in 1691-92 but he never published this work during his lifetime. Johann's lectures on the differential calculus were finally issued in 1923 as *Lectiones de calculo differentialium* (edited by Paul Schafheitlin and published in Basel), and it is from these notes that we are able to conclude definitely that the young Johann Bernoulli taught L'Hospital much of what he wrote in his differential calculus text in 1696.

Paris mathematicians engaged in a huge rivalry at that time. The math community constantly posed and solved problems, and the mathematician -- amateur or professional -- who solved the problem first earned great recognition and acclaim. However, contemporaries of L'Hospital describe him as an amiable young man, free of the rivalry that consumed much of seventeenth-century mathematics.

Probably the best mathematician in his circle, L'Hospital gained recognition for being the first to solve several problems. He wanted to know more about this new calculus of Leibniz and asked Johann Bernoulli -- then 24 years old, newly married, and without either a university affiliation or much money -- to help him interpret Leibniz. L'Hospital wrote the following letter, copyrighted in 1955, to Johann Bernoulli in Basel on March 17, 1694:

> I shall give you with pleasure a pension of three hundred livres, which will begin on the first of January of the present year, and I shall send two hundred livres for the first half of the year because of the journals you have sent, and it will be one hundred and fifty livres for the other half of the year, and so in the future. I promise to increase this pension soon, since I know it to be very moderate, and I shall do this as soon as my affairs are a little less confused….I am not so unreasonable as to ask for this all your time, but I shall ask you to give me occasionally some hours of your time to work on what I shall ask you-and also to communicate to me your discoveries, with the request not to mention them to others. I also ask you to send neither to M. Varignon nor to others copies of the notes that you have let me have, for it would not please me if they were made public. Send me your answer to all this and believe me, *Monsieur tout à vous.*
>
> *Le M. de Lhospital*

During 1691-1692, Johann Bernoulli spent four months tutoring L'Hospital both in Paris and at L'Hospital's country home in Oucques. In 1695 L'Hospital helped the young Bernoulli obtain a university job at the University of Groningen in the Netherlands, ending Bernoulli's money worries forever.

Analysis of Infinitely Small Quantities for the Understanding of Curves

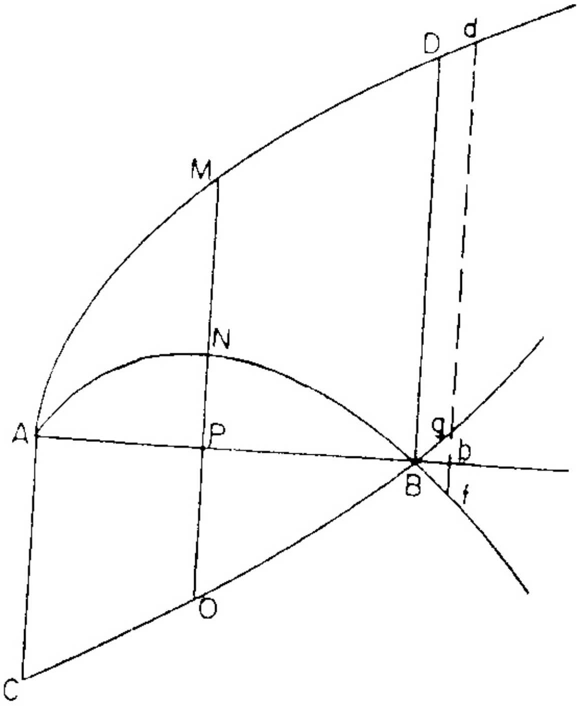
The year 1696 was a momentous year in the history of calculus. That year marked the publication of *Analyse des Infiniment Petits, Pour l'intelligence des lignes courbes* (Analysis of Infinitely Small Quantities for the Understanding of Curves), the first differential calculus textbook. The author (L'Hospital) originally published the text anonymously.

In the early 1920’s, the publication of Bernoulli's notes showed much overlap with L'Hospital's text, leaving no doubt that Bernoulli is the true father of L'Hospital's rule.

The text, divided into 10 sections, offers an introduction to Leibniz's differential calculus. In section 1, L'Hospital defines a variable quantity as one that continually increases or decreases and calls the infinitely small quantity by which the variable changes the difference or differential. Section 1 also gives the rules for obtaining the differentials of sums, products, quotients, powers, and roots. L'Hospital devotes section 2 to the problem of tangency; in *Calculus: Single-Variable*, fourth edition (Deborah Hughes-Hallett et al), you can find a similar problem to the one that he solves -- see number 15 on p. 556. Section 3 considers problems of maxima and minima, and section 4 deals with higher-order derivatives, known then as second and higher differences. Section 9 has caused the most discussion since 1696. It considers the indeterminate $\frac{0}{0}$.

Included below are several of L'Hospital's examples from section 9, translated by Edmund Stone and included in Dirk J. Struik's *A Source Book in Mathematics, 1200-1800*, pp. 315-316.

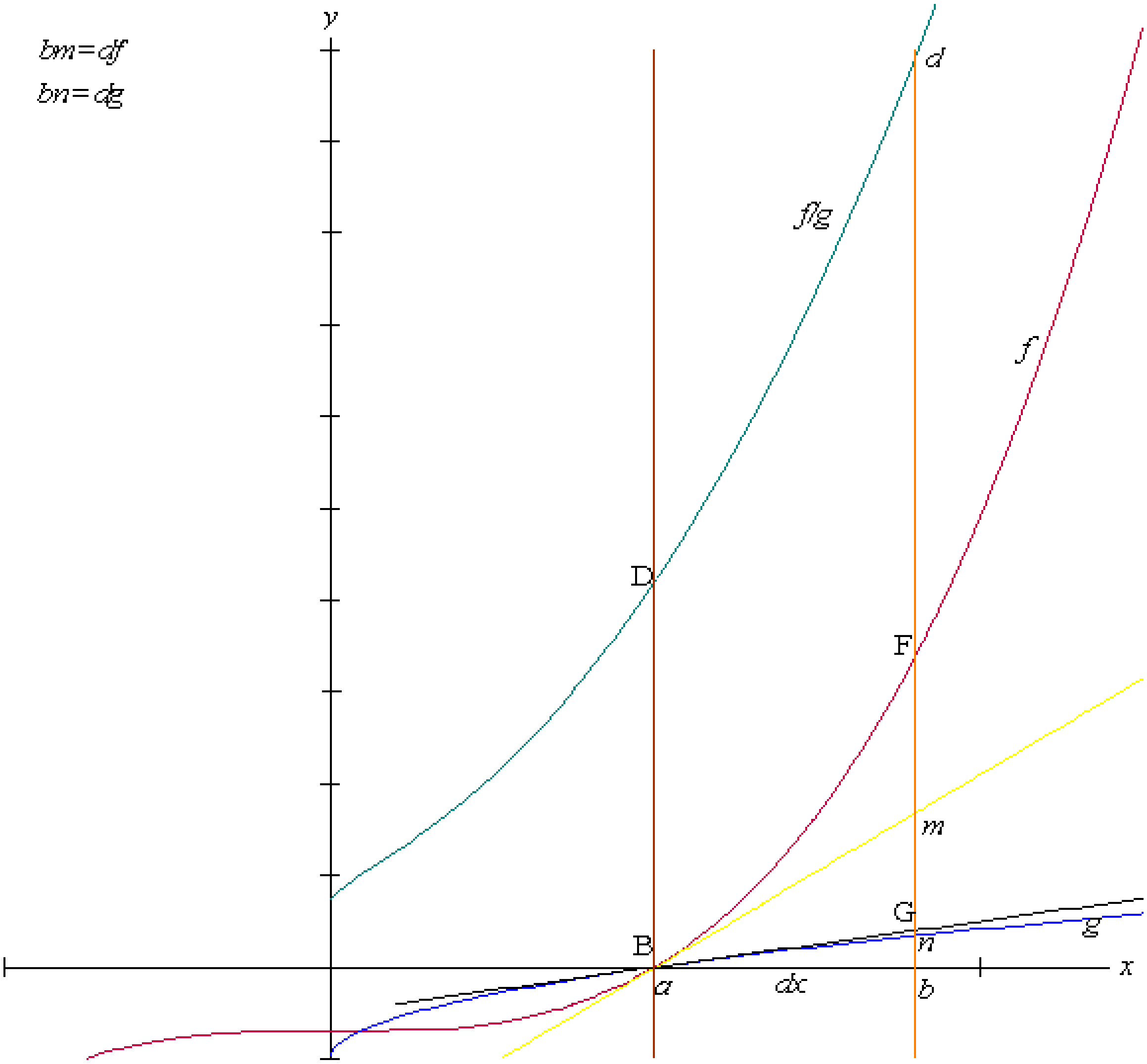
> Section IX. THE SOLUTION OF SOME PROBLEMS
> DEPENDING ON THE METHODS AFOREGOING
>
> 163. Proposition I. Let *AMD* be a curve (*AP* = *x*, *PM* = *y*, *AB* = *a*) of such a nature, that the value of the ordinate *y* is expressed by a fraction, the numerator and denominator of which, do each of them become 0 when *x* = *a*, viz. when the point *P* coincides with the given point *B*. It is required to find what will then be the value of the ordinate *BD*.



Graph reproduced from A Source Book in Mathematics, 1200-1800*, edited by Dirk J. Struik (Cambridge, Massachusetts: Harvard University Press, 1969), pp. 315-316; copyright 1969.*

In effect, L'Hospital asks the reader to find *BD* using geometry. The curves *ANB* and *COB* each are zero at *B*, and the curve *AMD* represents the quotient of the curve *ANB* divided by *COB*. *BD* represents the ordinate of $\frac{ANB}{COB}$ at $x = B$. The proof is geometrical, as is much of L'Hospital's text, and could be difficult for the modern reader to understand.

Below is a diagram that, I hope, will make L'Hospital's Rule easier to understand for the modern reader.



Graph made by Judy Broadwin.

We have $\frac{f(a)}{g(a)} = BD \approx bd = \frac{bF}{bG} \approx \frac{bm}{bn} = \frac{df}{dg} = \frac{df/dx}{dg/dx} = \frac{f'(a)}{g'(a)}$, where we have renamed the differential *bm* as *df* and the differential *bn* as *dg*.

In 1992, at the first TICAP (Technology Intensive Calculus for Advanced Placement) conference, Thomas Dick presented this local linearity interpretation of L'Hospital's rule to an audience of more than 300 mathematicians. The audience gasped, and Bert Waits called out, "That was what Bernoulli must have seen!" L'Hospital proved his rule using local linearity, and we had lost that interpretation for a very long time.

L'Hospital's text gives only two indeterminate-form examples of the type $\frac{0}{0}$. (The term **limit** did not appear in mathematics until the early nineteenth century.) The first, found in a similar version in a letter to L'Hospital from Bernoulli written on July 22, 1694, can be paraphrased as follows: Example I. 164. Let $y=\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}}$ and let $x = a$.

To evaluate y when $x = a$, L'Hospital takes the differential of the numerator evaluated at $x = a$ and divides it by the differential of the denominator at $x = a$, obtaining the result $BD=\frac{16}{9}a$.

The second and only other $\frac{0}{0}$ example in the L'Hospital text can be paraphrased as follows: Example II. 165. Let $y=\frac{aa-ax}{a-\sqrt{ax}}$. The reader is asked to find y when $x=a$.

L'Hospital solves this problem without calculus, in effect by multiplying the numerator and denominator of the fraction by $a+\sqrt{ax}$ and arriving at the result that $y = 2a$ when $x=a$. He was worried about this example and communicated his concern to Bernoulli in an early letter. (Note: This was not the second example in the Bernoulli letter. The example sent to L'Hospital on July 22, 1694, was $y=\frac{a\sqrt{ax}-xx}{a-\sqrt{ax}}$ for $x=a$. Then $y=3a$.)

The True Author

After publication of the textbook in 1696, Johann Bernoulli wrote to L'Hospital praising the work for its sound arrangements of propositions and for the intelligibility of the exposition. L'Hospital suggested to Bernoulli that they continue the collaboration to produce a book on the integral calculus, but Bernoulli wrote back that he was too busy at that time to work on another book.

The text was an immediate success. After his initial praise, Bernoulli, who was indebted to L'Hospital for his position at Groningen, started to claim credit for his contribution to the book. In 1698, he wrote to Leibniz complaining that L'Hospital had copied his work. After L'Hospital died in 1704, Bernoulli, feeling that he was free of his promise not to reveal his enormous contributions, started an all-out campaign to claim the text as his own. He practically accused L'Hospital of plagiarism, but it was not until 1955, with the publication of Bernoulli's early correspondence, that the mathematics community generally accepted Bernoulli as the true author.

The book is a wonderful text. Mostly geometrical, it contains many diagrams and letters and almost no equations. Using the text, contemporary mathematicians found it easy to interpret the new mathematics of the infinitely small. The text was reprinted in 1715. New French editions appeared in 1758, 1768, and 1781 and Latin translations arrived in 1764 and 1790. Newtonian-style calculus texts were published in England by Humphry Ditton in 1706, Charles Hayes in 1704, Thomas Simpson in 1737, and Colin Maclaurin in 1742. In the Leibniz tradition, the next major calculus text after L'Hospital's proved to be very famous: Maria Agnesi's book in 1748. Agnesi, like Leibniz, wrote of differentials and infinitesimals rather than Newton's fluxions. Agnesi, like Johann Bernoulli, used the symbol $\int y\,dx$ to represent the antiderivative, and only as an afterthought did she use the symbol to represent the area under a curve.

We accept today that the rule attributed to L'Hospital really belongs to Bernoulli, but L'Hospital wrote a text that kindled an enthusiasm and excitement for mathematics that lasted a long time and served as a model for future texts. The rule may indeed belong to Bernoulli, but L'Hospital gave it to mathematics.

Acknowledgement

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Judy Broadwin is currently teaching calculus and precalculus at Baruch College of the City University of New York. She taught AP Calculus at Jericho High School in Jericho, New York, for 29 years. She has been an AP Calculus Exam Reader, Table Leader, and Exam Leader (the latter for BC); she was a member of the AP Calculus Development Committee from 1987 to 1991. She is coauthor of the book Solutions: AP* Calculus Problems Part II, AB and BC, 1987-2001. *Broadwin has presented AP Calculus workshops and summer institutes for over 25 years. She is a winner of the Presidential Award for Excellence in Mathematics and Science Teaching and a Tandy Technology Scholar.*