Question 6

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer to part (a) to find $\lim_{x \to 0} \frac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\frac{1}{200}$.

Solution	Scoring
(a) $e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$ $= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$	3 : { 1 : two of $1, -x^2, \frac{x^4}{2}, -\frac{x^6}{6}$; 1 : remaining terms; 1 : general term }
(b) $\frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$ Thus, $\lim_{x \to 0} \left(\frac{1 - x^2 - f(x)}{x^4} \right) = -\frac{1}{2}$.	1 : answer
(c) $\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots \right) dt$ $= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$ Using the first two terms of this series, we estimate that $\int_0^{1/2} e^{-t^2}\,dt \approx \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = \frac{11}{24}$.	3 : { 1 : two terms; 1 : remaining terms; 1 : estimate }
(d) $\left\lvert \int_0^{1/2} e^{-t^2}\,dt - \frac{11}{24} \right\rvert < \left(\frac{1}{2}\right)^5 \cdot \frac{1}{10} = \frac{1}{320} < \frac{1}{200}$, since $\int_0^{1/2} e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.	2 : { 1 : uses the third term as the error bound; 1 : explanation }

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6A 1

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Work for problem 6(a)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x^2} = 1 + \frac{-x^2}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-1)^n x^{2n}}{n!} + \dots$$

Work for problem 6(b)

$$\lim_{x \to 0} \frac{\cancel{1} - \cancel{x^2} - \cancel{1} + \cancel{x^2} - \frac{x^4}{2!} + \frac{x^6}{3!} + \dots}{x^4} = \lim_{x \to 0} -\frac{1}{2} + \frac{x^2}{3!} + \dots = \boxed{-\frac{1}{2}}$$

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Work for problem 6(c)

$$\int_0^x e^{-t^2}\,dt = 0 + x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} + \ldots$$

$$\int_0^{1/2} e^{-t^2}\,dt \approx \frac{1}{2} - \frac{1}{24} = \boxed{\frac{11}{24}}$$

$$\lim_{n\to\infty}\left|\frac{x^2\cancel{x^{2n+1}}}{(2n+3)(n+1)\cancel{n!}}\cdot\frac{\cancel{(2n+1)}\cancel{n!}}{\cancel{x^{2n+1}}}\right| = 0 < 1$$

it is converges for all x (including ½)

∞

Work for problem 6(d)

the taylor series for $\int_0^{1/2} e^{-t^2}\,dt$ is an alternating series

(1) nth term goes to 0 $\lim_{n\to\infty} \frac{(\frac{1}{2})^{2n+1}}{(2n+1)\,n!} = 0$ ✓

(2) $|a_n| > |a_{n+1}|$ $|a_n| = \frac{\frac{1}{2}^{(2n+1)}}{(2n+1)n!} > \frac{\frac{1}{2}^{2n+1}\cdot\frac{1}{4}}{(2n+3)(n+1)!} = |a_{n+1}|$ ✓

numerator decreased

denominator increased

$$|E| < |a_{n+1}| = \frac{(\frac{1}{2})^5}{5\cdot 2!} = \frac{1}{320} < \frac{1}{200}$$

$$\therefore |E| < \frac{1}{200}$$

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6B

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Work for problem 6(a)

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ replace x with $-x^2$

$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!}$ Gen term $= \frac{(-1)^n x^{2n}}{n!}$

1st ~~four~~ non zero terms $1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!}$

Work for problem 6(b)

$\lim_{x\to 0} f(0) = 1 \lim_{x \to 0} \frac{1 - x^2 - 1}{x^4} = \frac{0}{0} \overset{L}{=} \frac{-2x}{4x^3} \overset{L}{=} \frac{-2 -2}{12x^2}$

$\lim_{x\to 0} \frac{-4}{12x^2} = -\infty$

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6B2

Work for problem 6(c)

$$\int f(x) \approx x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^6}{36}$$

$$\int_0^{1/2} f(x) \approx \frac{1}{2} - \frac{(\frac{1}{2})^3}{3} - 0 - \frac{(0)^3}{3}$$

$$\approx \frac{1}{2} - \frac{(\frac{1}{2})^3}{3}$$

$$\approx \frac{1}{2} - \frac{\frac{1}{8}}{3}$$

$$\approx \frac{1}{2} - \frac{1}{24}$$

$$\approx 11/24$$

Work for problem 6(d)

The taylor series for $\int f(x)$ is alternating. the error is less than the n+1 term because Term n+1 < Term n and $\lim_{n \to \infty}$ of terms = 0 and as I said It's alternating.

So Error $\leq$ n+1 term

$|error| \leq$ 3rd term

$$\leq \frac{(\frac{1}{2})^5}{10}$$

$$|E_1| \leq \frac{\frac{1}{32}}{10}$$

$$\text{Error} \leq \frac{1}{320} \to \leq \frac{1}{200}$$

Since error is less than $\frac{1}{320}$, which is less than 1/200 our aproximation will not be off by more than 1/200.

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6C1

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Work for problem 6(a)

$x=0$

0	e^{-x^2}	1	x^0	0!
1	$-2xe^{-x^2}$	1	x^1	1!
2	$4x^2e^{-x^2}$	1	x^2	2!
3	$-8x^3e^{-x^2}$	1	x^3	3!
4	$16x^4e^{-x^2}$	1	x^4	4!

$$\frac{f^n(c)(x-c)^n}{n!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{-x^2} = 1 - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!}$$

$$f(x) = 1 - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!}$$

Work for problem 6(b)

$$\cancel{x = x^2} \ \cancel{-x} \ \cancel{+x^2} - \frac{x^3}{2!} + \frac{x^4}{3!}$$

$$\lim_{x\to 0} \frac{1 - x^2 - \left(1 - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!}\right)}{x^4}$$

$$\lim_{x\to 0} \frac{-\frac{x^3}{2} + \frac{x^4}{6}}{x^4} = \lim_{x\to 0} -\frac{1}{2}\cdot\frac{x^3}{x^4} + \frac{1}{6}\cdot\frac{x^4}{x^4}$$

$$\lim_{x\to 0} -\frac{1}{2x} + \frac{1}{6} = \text{DNE}$$

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Continue problem 6 on page 15.

6C2

Work for problem 6(c)

$$e^{-x^2} = 1 - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!}$$

$$\int_0^x e^{-t^2} = x - \frac{x^3}{3} + \frac{x^4}{4 \cdot 2!} - \frac{x^5}{5 \cdot 3!}$$

$$\int_0^{1/2} e^{-t^2}\,dt = \left[x - \frac{x^3}{3}\right]_0^{1/2} = \frac{1}{2} - \frac{(1/2)^3}{3} = \frac{1}{2} - \frac{\frac{1}{8}}{\frac{3}{1}}$$

$$\frac{1 \cdot 12}{2 \cdot 12}\ \frac{1}{24} = \frac{11}{24}$$

Work for problem 6(d)

The next term in the series is used to find the error in a taylor polynomial. For this one it is $\frac{x^4}{4 \cdot 2!}$. By plugging in (0.1) for x, the error is a value less than $1/200$.

$$\frac{(0.1)^4}{8} < 1/200$$

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Question 6

Overview

This problem dealt with Taylor series. Part (a) assessed students' abilities to find the first four nonzero terms and the general term of the Taylor series for $f(x) = e^{-x^2}$. Although it would be possible to do this by computing derivatives of the function f, it was expected that students would start with the known Taylor series for the exponential function and use substitution. Part (b) asked for a limit of an indeterminate form $\left(\frac{0}{0}\right)$ involving the function f. Students were asked to use their answer about the Taylor series for f rather than using repeated applications of L'Hospital's Rule. Part (c) required students to formally manipulate the Taylor series for f in a way that could be used to estimate the value of a definite integral. Part (d) asked students to explain why the value of the estimate differed from the actual value of the definite integral by less than $\frac{1}{200}$. This question tested whether students could correctly use and justify the error bound for an alternating series whose terms are decreasing in absolute value to zero.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student has the first four nonzero terms and the correct general term so the first 3 points were earned. In part (b) none of the student's work earned any points. In part (c) the student makes an error in antidifferentiating the last term so only the first point for two terms was earned. The estimation is correct and earned the third point. In part (d) the student uses the third term as an error bound and successfully calculates $\frac{1}{320}$, and thus the first point was earned. The second point was not earned since the student explains that this is an alternating series but does not observe that the individual terms decrease in absolute value to 0.

Sample: 6C
Score: 4

The student earned 4 points: 1 point in part (a), no points in part (b), 3 points in part (c), and no points in part (d). In part (a) the student earned the first point for $1 - x^2$. The other terms are incorrect, and there is no general term so no other points were earned. In part (b), since the student's limit does not exist, the student was not eligible for the point. In part (c) the student correctly antidifferentiates the polynomial from part (a) so earned the first 2 points. The student's correct estimation earned the third point. In part (d) none of the student's work earned any points.