Question 4

Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

(b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x \ln x$.

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int \left(x^2 \ln x\right) dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = \int \left(x^2\right) dx = \frac{1}{3}x^3$$

Therefore,

$$\begin{aligned} f(x) &= \int \left(x^2 \ln x\right) dx \\ &= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C. \end{aligned}$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3} \ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

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Work for problem 4(a)

$f'(x) = x^2 \ln x$

$f'(e) = e^2 \ln(e) = e^2$

$\boxed{y - 2 = e^2(x-e)}$

Work for problem 4(b)

$f'(x) = x^2 \ln x$

$f''(x) = (x^2)(\frac{1}{x}) + (2x)\ln x = x + 2x\ln x$

$f''(1) = 1 + 2(1)\ln(1) = 1 + 0 = 1$

$f''(2) = 2 + 2(2)\ln(2) = 2 + 4\ln(2)$

$f''(3) = 3 + 2(3)\ln(3) = 3 + 6\ln(3)$

f is concave up on $1 < x < 3$ because f'' is positive on this interval

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NO CALCULATOR ALLOWED

Work for problem 4(c)

$f'(x) = x^2 \ln x$

let $u = \ln x$, $dv = x^2\,dx$

$du = \frac{1}{x}\,dx$ $v = \frac{1}{3}x^3$

$f(x) = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2\,dx$

$f(x) = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

$f(e) = 2$

$2 = \frac{1}{3}(e)^3 \ln(e) - \frac{1}{9}(e)^3 + C$

$2 = \frac{e^3}{3} - \frac{e^3}{9} + C$

$\frac{3e^3}{9} - \frac{e^3}{9} = \frac{2e^3}{9}$

$2 = \frac{2e^3}{9} + C$

$2 - \frac{2e^3}{9} = C$

$\frac{18}{9} - \frac{2e^3}{9} = C$

$C = \frac{18 - 2e^3}{9}$

$f(x) = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + \left(\frac{18 - 2e^3}{9}\right)$

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NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

$f'(x) = x^2 \ln x$

$f'(e) = e^2$

$y - 2 = e^2(x - e)$

$y = e^2 x - e^3 + 2$

Work for problem 4(b)

$f''(x) = x^2\left(\frac{1}{x}\right) + \ln x(2x)$

$= x + 2x\ln x = 0$

$x(1+2\ln x) = 0$

$x = 0, \quad \ln x = -1/2$

$x = e^{-1/2} = \frac{1}{e^{1/2}}$

$f''(x) = x(1+2\ln x)$

$f(x)$	1	3
$f''(x)$		

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NO CALCULATOR ALLOWED

Work for problem 4(c)

$f'(x) = x^2 \ln x$

$u = \ln x \quad dv = x^2 dx$

$du = \frac{1}{x} dx \quad v = \frac{x^3}{3}$

$\frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx$

$\frac{x^3}{3} \ln x - \frac{x^3}{9}$

$f(x) = \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right)$

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NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

$m = f'(e) = e^2$

$e = 2e^2 + b$

$b = e - 2e^2$

$\boxed{y = e^2 x + (e - 2e^2)}$

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Work for problem 4(b)

$f''(x) = \ln(x) \cdot 2x + x$

$\ln(2) \cdot 4 + 2 \Rightarrow +$

concave up, b/c $f'(2)$ is positive

Continue problem 4 on page 1

Work for problem 4(c)

$f(x) = \int x^2 \ln(x)\,dx$

Tabular Integration

x^2	+	$\ln x$
$2x$	−	$1/x$
2	+	$-2/x^2$
0		$+2/x^3$

$$f(x) = \frac{x^2}{x} + \frac{2x \cdot 2}{x^2} + \frac{2 \cdot 3}{x^3} + C$$

$$f(x) = x + 4/x + 6/x^3 + C$$

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Question 4

Overview

This problem presented students with the derivative of a function and an initial value. In part (a) students had to use the given information to write an equation for the tangent line at the initial value. In part (b) they needed to determine whether the graph of the function was concave up or concave down on a given interval and to justify their answer. Part (c) asked students to use antidifferentiation to find an explicit formula for the function. This involved using integration by parts, a BC-only topic.

Sample: 4A
Score: 9

The student earned all 9 points. Note that in part (b) the student includes the necessary reference to the interval $1 < x < 3$ when discussing the concavity of the graph of f.

Sample: 4B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In part (a) the student has a correct numerical value for the slope and uses it to determine an equation of the tangent line. In part (b) $f''(x) = x + 2x \ln x$ is correct, and the first 2 points were earned. The third point was not earned because there is no comment about the concavity of the graph of f. In part (c) the student correctly uses integration by parts and earned 2 points. Since there is no substitution for $f(e) = 2$ or a final solution for $f(x)$, the last 2 points were not earned. The missing $+C$ does not affect the antiderivative points but was required for the remaining points.

Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a) and 2 points in part (b). In part (a) the student earned the point for a correct $f'(e)$. The student's equation of the tangent line is incorrect. In part (b) $f''(x) = x + 2x \ln x$ is correct, and the first 2 points were earned. The explanation is not sufficient since the student states that $f'(2)$ is positive instead of $f''(2)$ is positive. For this problem, students were allowed to evaluate f' at only two points to determine whether f' is increasing or decreasing on the interval $1 < x < 3$. In part (c) the student's use of integration by parts is incorrect.