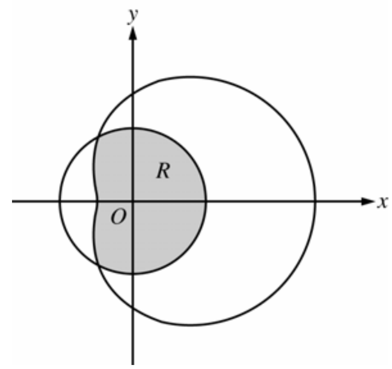
AP® CALCULUS BC
2007 SCORING GUIDELINES

Question 3

The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \dfrac{2\pi}{3}$ and $\theta = \dfrac{4\pi}{3}$.



(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\dfrac{dr}{dt} = \dfrac{dr}{d\theta}$. Find the value of $\dfrac{dr}{dt}$ at $\theta = \dfrac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\dfrac{dy}{dt} = \dfrac{dy}{d\theta}$. Find the value of $\dfrac{dy}{dt}$ at $\theta = \dfrac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) Area $= \dfrac{2}{3}\pi(2)^2 + \dfrac{1}{2}\displaystyle\int_{2\pi/3}^{4\pi/3} (3 + 2\cos\theta)^2 \, d\theta$

$= 10.370$

4 :
- 1 : area of circular sector
- 2 : integral for section of limaçon
  - 1 : integrand
  - 1 : limits and constant
- 1 : answer

(b) $\left.\dfrac{dr}{dt}\right|_{\theta=\pi/3} = \left.\dfrac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

The particle is moving closer to the origin, since $\dfrac{dr}{dt} < 0$ and $r > 0$ when $\theta = \dfrac{\pi}{3}$.

2 :
- 1 : $\left.\dfrac{dr}{dt}\right|_{\theta=\pi/3}$
- 1 : interpretation

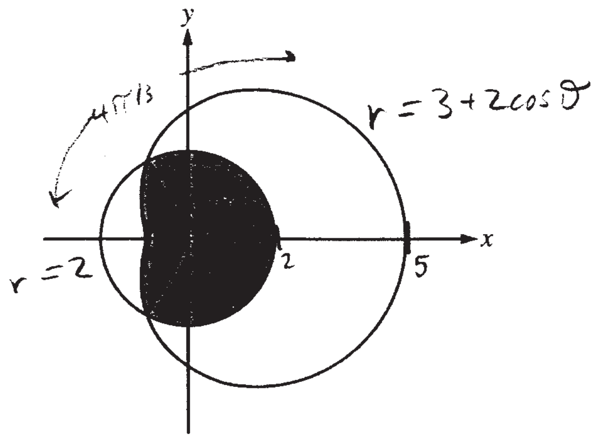
(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$

$\left.\dfrac{dy}{dt}\right|_{\theta=\pi/3} = \left.\dfrac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

The particle is moving away from the x-axis, since $\dfrac{dy}{dt} > 0$ and $y > 0$ when $\theta = \dfrac{\pi}{3}$.

3 :
- 1 : expression for y in terms of θ
- 1 : $\left.\dfrac{dy}{dt}\right|_{\theta=\pi/3}$
- 1: interpretation

3A



Work for problem 3(a)

$$\text{Area} = \frac{1}{2}\int_{-2\pi/3}^{2\pi/3} (2)^2\, d\theta + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3+2\cos\theta)^2\, d\theta$$

$$\approx 10.370$$

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Continue problem 3 on page 9.

3A$_2$

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Work for problem 3(b)

$t = \theta = \pi/3$

$\frac{dr}{dt} = \frac{d}{dt}\left[3 + 2\cos t\right] = -2\sin t$

$\left.\frac{dr}{dt}\right|_{t=\pi/3} = -2\sin\frac{\pi}{3} = -\sqrt{3}$

At time $t = \frac{\pi}{3}$, The particle is approaching the origin at a rate of $\sqrt{3}$.

Work for problem 3(c)

$t = \theta = \pi/3$

Since $r = 3 + 2\cos\theta$,

$y = r\sin\theta = (3 + 2\cos\theta)(\sin\theta)$

$y(t) = (3 + 2\cos t)(\sin t)$

$\frac{dy}{dt} = 4\cos^2 t + 3\cos t - 2$

$\left.\frac{dy}{dt}\right|_{t=\frac{\pi}{3}} = 4\cos^2\left(\frac{\pi}{3}\right) + 3\cos\left(\frac{\pi}{3}\right) - 2 = \boxed{\frac{1}{2}}$

At time $t = \pi/3$, the particle's height is increasing at a rate of $1/2$. Thus, the particle is moving Up at a rate of $1/2$ (but it might also be moving horizontally).

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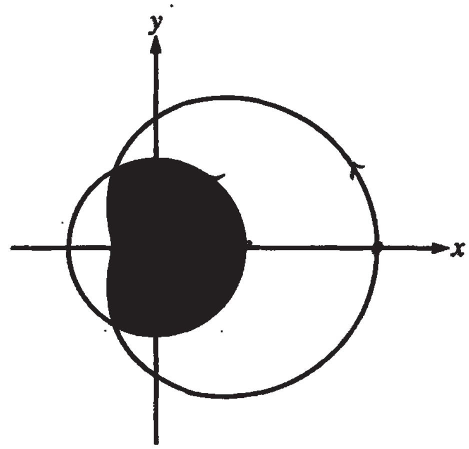
END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.



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Work for problem 3(a)

Let $r_1 = 2 \quad r_2 = 3+2\cos\theta$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$A = \frac{1}{2}\int_0^{2\pi/3}(r_1)^2\,d\theta + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(r_2)^2\,d\theta + \frac{1}{2}\int_{4\pi/3}^{2\pi}(r_1)^2\,d\theta =$$

$$= \frac{1}{2}\left(8.37758 + 3.985787 + 8.37758\right)$$

$$= 10.37047$$

Continue problem 3 on page 9.

Work for problem 3(b)

$(x(t), y(t))$ $\theta = 0$ when $t = 0$

$r = x\cos\theta$
$r = y\sin\theta$

$\frac{dr}{dt} = \frac{dr}{d\theta}$, $r = 3 + 2\cos\theta$

$\frac{dr}{dt} = \frac{dr}{d\theta} = -2\sin\theta$

$\left.\frac{dr}{dt}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{3}} = -2\sin\frac{\pi}{3} = -\sqrt{3}$

Particle is moving down and to the right @ $\theta = \frac{\pi}{3}$

$\frac{dr}{d\theta} = \frac{-dx}{d\theta}\sin\theta$
$-\sqrt{3} = -\frac{dx}{d\theta}\left(\frac{\sqrt{3}}{2}\right)$
$\frac{dr}{d\theta} = \frac{dy}{d\theta}\cos\theta$
$-\sqrt{3} = \frac{1}{2}\frac{dy}{d\theta}$
$\frac{dx}{d\theta} = 2$
$\frac{dy}{d\theta} = -2\sqrt{3}$

$r^2 = x^2 + y^2$

Work for problem 3(c)

$\frac{dy}{dt} = \frac{dy}{d\theta}$

$y = r\sin\theta$
$y = (3 + 2\cos\theta)\sin\theta$
$\frac{dy}{d\theta} = -2\sin\theta\sin\theta + \cos\theta(3 + 2\cos\theta)$
$\frac{dy}{d\theta} = -2\sin^2\theta + 3\cos\theta + 2\cos^2\theta$

$\left.\frac{dy}{d\theta}\right|_{\theta=\frac{\pi}{3}} = -2\left(\frac{\sqrt{3}}{2}\right)^2 + 3\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2$

$= \frac{-3}{2} + \frac{3}{2} + \frac{1}{4} = \boxed{\frac{1}{4}}$

particle

END OF PART A OF SECTION II
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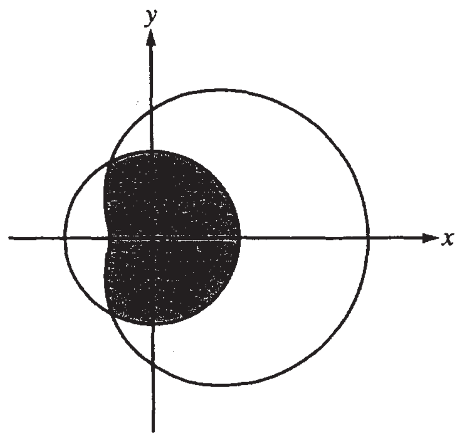
3C₁



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Work for problem 3(a)

$$\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2)^2 - (3+2\cos\theta)^2 \, d\theta = 2.196$$

Continue problem 3 on page

3C2

Work for problem 3(b)

$y = r\sin\theta$

$x = r\cos\theta$

$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[(3+2\cos\theta)\sin\theta\right]$

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Work for problem 3(c)

$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = r\cos\theta + \sin\theta \cdot \frac{dr}{d\theta} = (3+2\cos\theta)\cos\theta + \sin\theta(-2\sin\theta)$

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END OF PART A OF SECTION II

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Question 3

Overview

This problem presented students with two curves described in polar coordinates with r as a function of θ. The values of θ for which the two curves intersect were also given. Part (a) judged students' ability to find the area of a region bounded by curves described in polar coordinates. Parts (b) and (c) involved the behavior of a particle moving with nonzero velocity along one of the polar curves (and with constant angular velocity $\frac{d\theta}{dt} = 1$, although students did not need to know that to answer the questions). Students were asked to compute $\frac{dr}{dt}$ and $\frac{dy}{dt}$ at a specific value of θ, and then to interpret their answers in terms of the motion of this particle. The interesting aspect of the motion was that at this instant, the distance of the particle from the origin was decreasing while its vertical distance from the x-axis was increasing.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 4 points in part (a), 1 point in part (b), and 1 point in part (c). The student earned all 4 points in part (a). The student correctly evaluates the derivative at $\theta = \frac{\pi}{3}$ and earned the first point in part (b), but the student does not interpret this value as the particle moving closer to the origin so the second point was not earned. In part (c) the student correctly states $y = r\sin\theta$ and earned the first point. The student incorrectly evaluates the derivative in part (c) and then fails to give an interpretation of the value of the derivative.

Sample: 3C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), and 1 point in part (c). The student did not earn the first point for the circular area, but the integrand and the limits and constant points were earned for a section of the limaçon relevant to the problem. The student did not earn the answer point. In part (b) the student does not evaluate the derivative and so was not eligible for the interpretation point. In part (c) the student states the correct form of the derivative of $y = r\sin\theta$ and earned the first point. The student does not evaluate the derivative at $\theta = \frac{\pi}{3}$ and so did not earn the second point. The student was not eligible for the answer point.