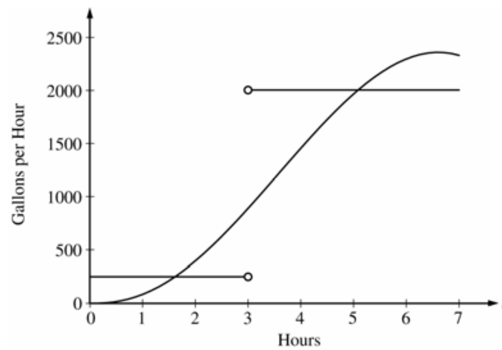
Question 2

The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin\left(\sqrt{t}\right)$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is

$$g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases} \text{ gallons per hour.}$$



The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

(a) $\int_0^7 f(t)\,dt \approx 8264$ gallons

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

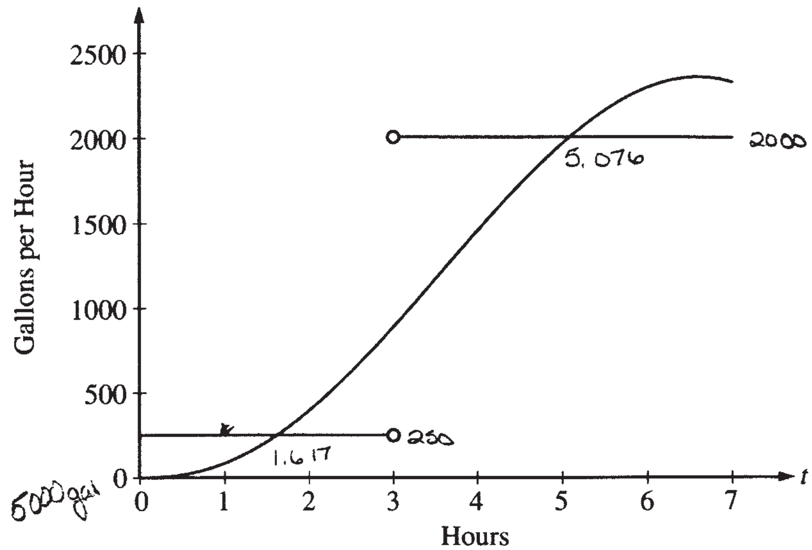
$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0,\ 3,$ and 7.

t (hours)	gallons of water
0	5000
3	$5000 + \int_0^3 f(t)\,dt - 250(3) = 5126.591$
7	$5126.591 + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

$5 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{integrand} \\ 1 : \text{amount of water at } t = 3 \\ 1 : \text{amount of water at } t = 7 \\ 1 : \text{conclusion} \end{cases}$



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Work for problem 2(a)

$\int_0^7 f(t)dt$

8264 gallons

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Continue problem 2 on page 7.

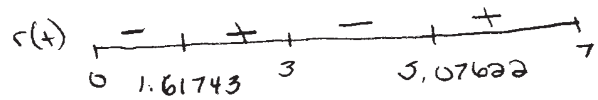
Work for problem 2(b)

$r(t) = f(t) - g(t)$

$r(t) = f(t) - 250$ for $0 \le t < 3$ $\quad$ $r(t) = f(t) - 2000$ for $3 < t \le 7$

$0 = f(t) - 250$ $\quad$ $0 = f(t) - 2000$

$t = 1.61743$ $\quad$ $t = 5.07622$



The amount of water in the tank is decreasing on $(0, 1.61743)$ b/c $r(t)$ that is defined for $0 \le t < 3$ is negative. It is also decreasing on $(3, 5.07622)$ b/c $r(t)$ that is defined for $3 < t \le 7$ is negative.

Work for problem 2(c)

Relative max at time $t=3$ b/c $r(t)$ changes from $(+)$ to $(-)$

$$5000 + \int_0^3 r(t)\,dt =$$

5126.591

$(0, 5000)$

$(3, 5126.591)$

$(7, 4513.807)$

$$5000 + \int_0^3 r(t)\,dt + \int_3^7 r(t)\,dt =$$

4513.807

The amount of water in the tank is greatest at time $t=3$ which is a relative max for the function of the amount of water in the tank. At the boundaries of the interval the amount of water in the tank is equal to 5000, at $t=0$, and 4514, at $t=7$. At $t=3$, the amount of water in the tank is 5127 gallons, which is greater than the values at the boundaries.

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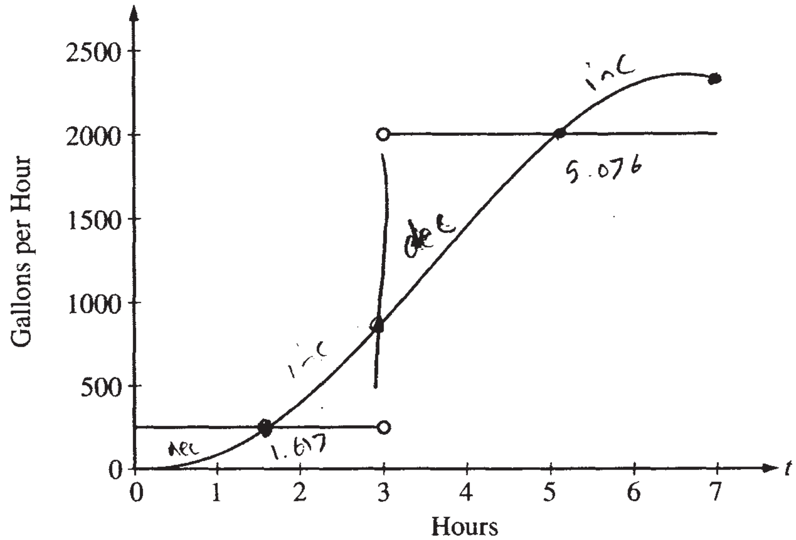

t=0, 5000 gals

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Work for problem 2(a)

a) $\int_0^7 100t^2 \sin(\sqrt{t})\,dt =$ 8264 gallons

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Continue problem 2 on page 7.

Work for problem 2(b)

b) water is decreasing $(0, 1.617)$, because $g(t) > f(t)$ on that interval.

water is decreasing $(3, 5.076)$, because $g(t) > f(t)$ on that interval

Work for problem 2(c)

$t=0$, 5000 gals

$h(0)=5000$ $h(7)=-736.19$

$h(3) = -123.409$

c) $h(t) = f(t) - g(t)$

$h(7)=f(7) - g(7)$

$100(7)^2 \sin\sqrt{7} - 2000$

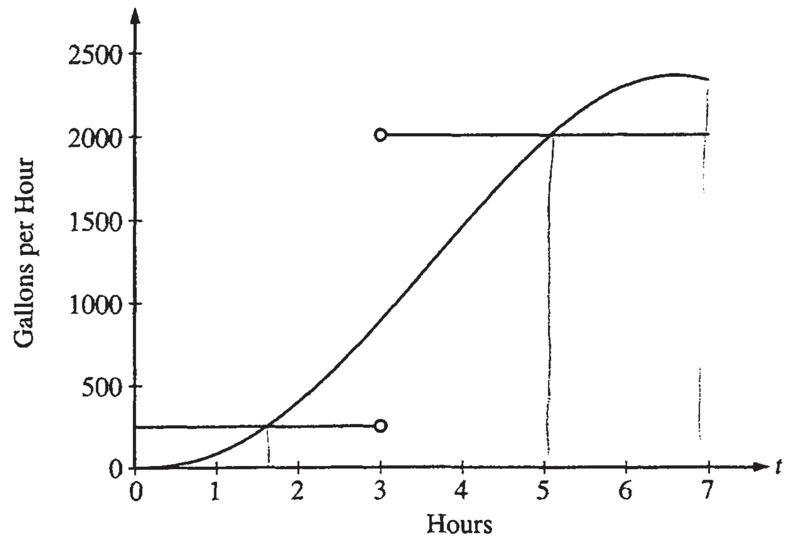
$5000 + \int_0^7 100t^2 \sin\sqrt{t} - 2000 \, dt = -736.19$

$5000 + \int_0^0 100t^2 \sin\sqrt{t} - 2000 \, dt = 5000$

$5000 + \int_0^3 100t^2 \sin\sqrt{t} - 2000 \, dt = -123.409$

At $t=0$, the amount of water is greatest

At $t=0$, the amount of water is 5000 gallons



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Work for problem 2(a)

$$\int_0^7 100t^2 \sin(\sqrt{t})\,dt \approx 8264 \text{ gallons}$$

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Work for problem 2(b)

The amount in the tank is decreasing on the intervals (0, 1.617) ∪ (1.617, 5.076) because on those intervals the rate of water entering is less than the rate of the water leaving.

Work for problem 2(c)

At time t=7. because the water entering is greater than the water leaving is at a max for the interval.

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Question 2

Overview

This problem presented students with two functions that modeled the rates, in gallons per hour, at which water entered and left a storage tank. The latter function was piecewise-constant. Graphs of each function were also provided. In part (a) students had to use a definite integral to find the total amount of water that entered the tank over a given time interval. Part (b) measured their abilities to compare the two rates to find, with justification, the time intervals during which the amount of water in the tank was decreasing. This could be determined directly from the graphs and the information given about the points of intersection, but students needed to be able to handle the point of discontinuity in the piecewise-defined function. Part (c) asked for the time at which the amount of water was at an absolute maximum and the value of this maximum amount to the nearest gallon. Again, dealing with the critical point at the discontinuity was an important part of the analysis, as was using the net rate of change during the first three hours and during the last four hours to compute the total amount of water in the tank at $t = 3$ and $t = 7$, respectively.

Sample: 2A
Score: 9

The student earned all 9 points. Note that in part (b), in the presence of the correct numerical values in reported intervals, errors in the use of open, closed, or half-open interval notation were ignored, and the student earned the interval point. The student defines $r(t)$ in part (b) and that definition may be used in part (c).

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In part (a) the student gives the correct integral and the correct answer and earned both the integral and answer points. In part (b) the student gives the correct intervals and a correct reason and earned both the interval and reason points. In part (c) the student considers $t = 0$, $t = 3$, and $t = 7$ as candidates for the absolute maximum with the evaluation of the three integrals presented. The student therefore considers $t = 3$ a candidate and earned the first point. The student presents the integrand $f(t) - g(t)$ in an integral and earned the second point. The incorrect use of the rule for $g(t)$ results in incorrect values for the amount of water at time $t = 3$ and $t = 7$, so the third and fourth points in part (c) were not earned; and the student presents an incorrect conclusion, so the fifth point was not earned.

Sample: 2C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), and no points in part (c). In part (a) the student gives the correct integral and the correct answer and earned both the integral and answer points. In part (b) the student reports an incorrect left endpoint on the second interval, so the interval point was not earned. The student provides correct reasoning and earned the reason point. In part (c) the student never considers $t = 3$ as a candidate for the absolute maximum, so the first point was not earned. No integrand in an integral is presented, and the amounts of water at $t = 3$ and $t = 7$ are not calculated. The second, third, and fourth points in part (c) were not earned, and the student presents an incorrect conclusion, so the fifth point was not earned.