AP® CALCULUS BC
2007 SCORING GUIDELINES (Form B)

Question 2

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \arctan\left(\frac{t}{1+t}\right) \text{ and } \frac{dy}{dt} = \ln\left(t^2 + 1\right)$$

for $t \geq 0$. At time $t = 0$, the object is at position $(-3, -4)$. (Note: $\tan^{-1} x = \arctan x$)

(a) Find the speed of the object at time $t = 4$.

(b) Find the total distance traveled by the object over the time interval $0 \leq t \leq 4$.

(c) Find $x(4)$.

(d) For $t > 0$, there is a point on the curve where the line tangent to the curve has slope 2. At what time t is the object at this point? Find the acceleration vector at this point.

(a) Speed $= \sqrt{x'(4)^2 + y'(4)^2} = 2.912$	1 : speed at $t = 4$
(b) Distance $= \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 6.423$	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(c) $x(4) = x(0) + \int_0^4 x'(t)\,dt$ $= -3 + 2.10794 = -0.892$	$3 : \begin{cases} 2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{uses } x(0) = -3 \end{cases} \\ 1 : \text{answer} \end{cases}$
(d) The slope is 2, so $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2$, or $\ln\left(t^2+1\right) = 2\arctan\left(\frac{t}{1+t}\right)$. Since $t > 0$, $t = 1.35766$. At this time, the acceleration is $\langle x''(t), y''(t)\rangle\big\vert_{t=1.35766} = \langle 0.135, 0.955\rangle$.	$3 : \begin{cases} 1 : \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2 \\ 1 : t\text{-value} \\ 1 : \text{values for } x'' \text{ and } y'' \end{cases}$

BC
2A1

Work for problem 2(a)

$$\text{speed} = \sqrt{\left[\arctan\left(\frac{t}{1+t}\right)\right]^2 + \left[\ln(t^2+1)\right]^2}$$

$$\text{speed}\Big|_{t=4} = \sqrt{0.45528 + 8.027}$$

$$= 2.912$$

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Work for problem 2(b)

$$\int_0^4 \sqrt{\left[\arctan\left(\frac{t}{1+t}\right)\right]^2 + \left[\ln(t^2+1)\right]^2}$$

$$= 6.4233$$

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Continue problem 2 on page

BC
2A₂

Work for problem 2(c)

$$\frac{dx}{dt} = \arctan\left(\frac{t}{1+t}\right)$$

$$\int dx = \int \arctan\left(\frac{t}{1+t}\right) dt$$

$$x(4) = x(0) + \int_0^4 \arctan\left(\frac{t}{1+t}\right) dt$$

$$= -3 + 2.1079$$

$$= -0.8921$$

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Work for problem 2(d)

$$\frac{dy}{dx} = \frac{\ln(t^2+1)}{\arctan\left(\frac{t}{1+t}\right)}$$

$$2 = \frac{\ln(t^2+1)}{\arctan\left(\frac{t}{1+t}\right)}$$

$$0 = \frac{\ln(t^2+1)}{\arctan\left(\frac{t}{1+t}\right)} - 2$$

$t = 1.3576631$

Let $1.3576631 = c$

$x''(c) = 0.13510$

$y''(c) = 0.955$

$\bar{a}(1.358) = \langle 0.1351, 0.955 \rangle$

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BC
2B1

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Work for problem 2(a)

The speed $v(t)$ is given by

$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\therefore v(4) = \sqrt{[\arctan(0.8)]^2 + [\ln(17)]^2} \approx 2.912$

Work for problem 2(b)

total distance $d(t)$ is given by

$d(t) = \int_0^t v(s)\,ds$

$\therefore d(4) = \int_0^4 \sqrt{\left[\arctan\left(\frac{t}{1+t}\right)\right]^2 + [\ln(t^2+1)]^2}\,dt \approx 6.423$

Continue problem 2 on page

BC
2B2

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Work for problem 2(c)

$x(t) = \int \arctan\left(\frac{t}{1+t}\right) dt$

$x(4) = \int_0^4 \arctan\left(\frac{t}{1+t}\right) dt \approx 2.108$

Work for problem 2(d)

When the slope of the tangent line is 2, $\frac{dy}{dx}\left(= \frac{dy}{dt} \Big/ \frac{dx}{dt}\right) = 2$.

So $\ln(t^2+1) / \arctan\left(\frac{t}{1+t}\right) = 2$, and $t \approx 1.358$.

And the acceleration vector $\vec{a}(t)$ is

$\vec{a}(1.358) = \frac{d^2x}{dt^2}_{t=1.358} \hat{x} + \frac{d^2y}{dt^2}_{t=1.358} \hat{y}$

$= 0.233\,\hat{x} + 0.782\,\hat{y}$.

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BC
2C, 1

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Work for problem 2(a)

~~Spee~~ Speed of an object $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ where $\frac{dx}{dt} = \arctan\left(\frac{t}{1+t}\right)$

$\frac{dy}{dt} = \ln(t^2+1)$.

Since $t = 4$, $\frac{dx}{dt} = 0.675$

$\frac{dy}{dt} = 2.833$.

$\therefore$ speed $= \sqrt{(0.675)^2 + (2.833)^2}$

$= 2.912$.

$\therefore$ speed $= 2.912$.

Work for problem 2(b)

total distance traveled $\int_a^b \frac{dy}{dx}$ where $\frac{dy}{dt}$ is velocity.

in this case, $\int_0^4 \frac{\ln(t^2+1)}{\tan^{-1}\left(\frac{t}{1+t}\right)}$ Since $\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$.

$\therefore$ using calculator,

total distance traveled in the time interval $0 \le t \le 4$

$= 9.953$

Continue problem 2 on page

Bc
2c₂

Work for problem 2(c)

$x(t) = \int \arctan\left(\frac{t}{1+t}\right) dt$

$= \int \arctan(t+1)\,.\,dt$

using calculator to integrate ~~and~~ substitute $t=4$,

$x(4) = 5.238.$

Work for problem 2(d)

the point the tangent line has slope 2 is when $\frac{dy}{dx} = 2$.

$\therefore \frac{\ln(t^2+1)}{\tan^{-1}\left(\frac{t}{1+t}\right)} = 2$, $t = -2.588$ or 1.358.

but since t cannot be negative.

$t = 1.358$

acceleration vector is $(x''(t), y''(t))$.

$x''(t) = \frac{d}{dt} \arctan\left(\frac{t}{1+t}\right)$, $y''(t) = \frac{d}{dt} \ln(t^2+1)$

acceleration vector

$\therefore \left(\frac{1}{2x^2+2x+1}, \frac{2x}{x^2+1}\right)$

$\left(\frac{1}{2x^2+2x+1}, \frac{2x}{x^2+1}\right)$

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AP® CALCULUS BC
2007 SCORING COMMENTARY (Form B)

Question 2

Sample: 2A
Score: 9

The student earned all 9 points.

Sample: 2B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 2 points in part (d). Correct work is presented in parts (a) and (b). In part (c) the first point was earned for the correct setup of the integral. The student does not use the initial condition that $x(0) = -3,$ so the last 2 points were not earned. In part (d) the student does not correctly evaluate the components of the acceleration vector, so the last point was not earned.

Sample: 2C
Score: 4

The student earned 4 points: 1 point in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d). Correct work is presented in part (a). In part (b) the student does not use the fact that the distance traveled is found by integrating the speed. In part (c) the first point was earned for a correct integrand. The student does not use the initial condition that $x(0) = -3,$ so the last 2 points were not earned. In part (d) the student was awarded the first 2 points. The first point was earned when the student sets $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2.$ The student does not find the acceleration vector at $t = 1.358,$ and so the third point was not earned. The student could have used the graphing calculator to determine the acceleration vector by the numerical derivative.