Question 6

Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

(c) For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

(a) $f'(x) = \dfrac{k}{2\sqrt{x}} - \dfrac{1}{x}$

$f''(x) = -\dfrac{1}{4}kx^{-3/2} + x^{-2}$

$2 : \begin{cases} 1 : f'(x) \\ 1 : f''(x) \end{cases}$

(b) $f'(1) = \dfrac{1}{2}k - 1 = 0 \Rightarrow k = 2$

When $k = 2$, $f'(1) = 0$ and $f''(1) = -\dfrac{1}{2} + 1 > 0$.

f has a relative minimum value at $x = 1$ by the Second Derivative Test.

$4 : \begin{cases} 1 : \text{sets } f'(1) = 0 \text{ or } f'(x) = 0 \\ 1 : \text{solves for } k \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(c) At this inflection point, $f''(x) = 0$ and $f(x) = 0$.

$f''(x) = 0 \Rightarrow \dfrac{-k}{4x^{3/2}} + \dfrac{1}{x^2} = 0 \Rightarrow k = \dfrac{4}{\sqrt{x}}$

$f(x) = 0 \Rightarrow k\sqrt{x} - \ln x = 0 \Rightarrow k = \dfrac{\ln x}{\sqrt{x}}$

Therefore, $\dfrac{4}{\sqrt{x}} = \dfrac{\ln x}{\sqrt{x}}$

$\Rightarrow 4 = \ln x$

$\Rightarrow x = e^4$

$\Rightarrow k = \dfrac{4}{e^2}$

$3 : \begin{cases} 1 : f''(x) = 0 \text{ or } f(x) = 0 \\ 1 : \text{equation in one variable} \\ 1 : \text{answer} \end{cases}$

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6A,

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Work for problem 6(a)

$f(x) = k\sqrt{x} - \ln x$

$f'(x) = \frac{k}{2}x^{-1/2} - \frac{1}{x}$

$f''(x) = \frac{-k}{4}x^{-3/2} + x^{-2}$

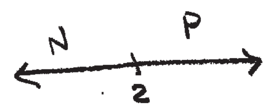
Work for problem 6(b)

$f'(x) = \frac{k}{2}x^{-1/2} - \frac{1}{x}$

$0 = \frac{k}{2}(1)^{-1/2} - \frac{1}{1}$

$1 = \frac{k}{2}$; $k = 2$

for $k = 2$, f has a critical point at $x = 1$.



For $k = 2$, f has a relative minimum at $x = 1$, since the derivative changes from negative to positive at that location, which means the function changes from decreasing to increasing, creating a local minimum

Continue problem 6 on page 15.

6A2

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Work for problem 6(c)

$f''(x) = \frac{-k}{4} x^{-3/2} + x^{-2}$

$0 = \frac{-k}{4} x^{-3/2} + x^{-2}$

$x^{-2} = \frac{k}{4} x^{-3/2}$

$4x^{-2} = k x^{-3/2}$

$4x^{-1/2} = k$

$0 = 4x^{-1/2}(x^{1/2}) - \ln x$

$0 = 4 - \ln x$

$4 = \ln x$

$x = e^4$

$(e^4)^{-2} = \frac{k}{4} (e^4)^{-3/2}$

$e^{-8} = \frac{k}{4} e^{-6}$

$e^{-2} = \frac{k}{4}$

$k = \frac{4}{e^2}$

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6B1

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Work for problem 6(a)

$f(x) = k(x)^{1/2} - \ln x$

$f'(x) = \dfrac{kx^{-1/2}}{2} - \dfrac{1}{x}$

$f''(x) = \dfrac{-kx^{-3/2}}{4} - \dfrac{-1}{x^2}$

$= \dfrac{-kx^{-3/2}}{4} + \dfrac{1}{x^2}$

Work for problem 6(b)

x	.5	1	2
$\frac{1}{\sqrt{.5}} - 2$		0	$\frac{1}{4}$
	−	0	+

$\frac{1}{2} - \frac{1}{4}$

$f(x) = 0 = \dfrac{kx^{-1/2}}{2} - \dfrac{1}{x}$

$f(1) = 0 = \dfrac{k \cdot 1}{2} - 1$

$1 = \dfrac{k}{2}$

$\boxed{2 = k}$

relitive minumum at $x = 1$ because the function is decreasing to the left of the critical point and increasing on the right.

Continue problem 6 on page 15.

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6B$_2$

Work for problem 6(c)

$$0 = \frac{-k x^{-3/2}}{4} + \frac{1}{x^2}$$

$$-\frac{1}{x^2} = \frac{-k x^{-3/2}}{4}$$

$$\frac{4}{-x^2} = -k x^{-3/2}$$

$$\frac{4}{x^2} = k x^{-3/2}$$

$$\frac{4x^{3/2}}{x^2} = k$$

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6C1

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Work for problem 6(a)

$kx^{\frac{1}{2}} - \ln x$

$f'(x) = \frac{1}{2}kx^{-1/2} - \frac{1}{x}$

$f''(x) = -\frac{1}{4}kx^{-3/2} + x^{-2}$

Work for problem 6(b)

critical point at $x=1$ what value of constant

$k(1)^{1/2} - \ln(1)$

$k\,1/2 =$

$\boxed{\text{relative minimum}}$

$k = \frac{f(x)}{1/2}$

Continue problem 6 on page 1:

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6C₂

Work for problem 6(c)

$$0 = -\frac{1}{4}kx^{-3/2} + x^{-2}$$

$$0 = -\frac{1}{4}k\cancel{x}^{-3/2} + \cancel{x}^{-2}$$

$$k = -\frac{1}{4}x^{-3/2} + x^{-2}$$

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Question 6

Overview

This problem presented students with a function that contained a parameter k. In part (a) students had to find the first and second derivatives of the function, making the distinction between the parameter and the variable. Parts (b) and (c) involved finding values of k so that the function or its graph would satisfy certain properties. In part (b) students had to find the value of k for which the function had a critical point at $x = 1$, and then determine whether the function had a relative minimum, relative maximum, or neither at this critical point. In part (c) students were told that the graph of the function had a point of inflection on the x-axis for a certain value of k and were asked to find that value. The x-coordinate of the point of inflection was not given so students had to write and then solve two nonlinear equations to determine the value of k (and possibly the value of x). Because the problem stated that a point of inflection existed, students were not required to justify that the k value they found actually produced a point of inflection of the graph of the function.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 3 points in part (b), and 1 point in part (c). In part (a) the student has the correct first and second derivatives, which earned the points. In part (b) the student earned the points for setting the first derivative equal to 0, solving for k, and declaring the critical point a minimum. The justification point was not earned because the student states that the function decreases and then increases after 1. In part (c) the student earned the point for setting the second derivative equal to 0.

Sample: 6C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (c). In part (a) the student has the correct first and second derivatives, which earned the points. In part (c) the student earned the point for setting the second derivative equal to 0.