## Question 4

A particle moves along the $x$-axis with position at time $t$ given by $x(t) = e^{-t} \sin t$ for $0 \le t \le 2\pi$.

(a) Find the time $t$ at which the particle is farthest to the left. Justify your answer.

(b) Find the value of the constant $A$ for which $x(t)$ satisfies the equation $Ax''(t) + x'(t) + x(t) = 0$ for $0 < t < 2\pi$.

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(a) $x'(t) = -e^{-t} \sin t + e^{-t} \cos t = e^{-t} (\cos t - \sin t)$

$x'(t) = 0$ when $\cos t = \sin t$. Therefore, $x'(t) = 0$ on $0 \le t \le 2\pi$ for $t = \frac{\pi}{4}$ and $t = \frac{5\pi}{4}$.

The candidates for the absolute minimum are at $t = 0, \frac{\pi}{4}, \frac{5\pi}{4}$, and $2\pi$.

| $t$ | $x(t)$ |
|---|---|
| $0$ | $e^0 \sin(0) = 0$ |
| $\frac{\pi}{4}$ | $e^{-\frac{\pi}{4}} \sin\left(\frac{\pi}{4}\right) > 0$ |
| $\frac{5\pi}{4}$ | $e^{-\frac{5\pi}{4}} \sin\left(\frac{5\pi}{4}\right) < 0$ |
| $2\pi$ | $e^{-2\pi} \sin(2\pi) = 0$ |

The particle is farthest to the left when $t = \frac{5\pi}{4}$.

$5 : \begin{cases} 2 : x'(t) \\ 1 : \text{sets } x'(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(b) $x''(t) = -e^{-t} (\cos t - \sin t) + e^{-t} (-\sin t - \cos t)$

$\quad = -2e^{-t} \cos t$

$Ax''(t) + x'(t) + x(t)$

$\quad = A\left(-2e^{-t} \cos t\right) + e^{-t} (\cos t - \sin t) + e^{-t} \sin t$

$\quad = (-2A + 1)e^{-t} \cos t$

$\quad = 0$

Therefore, $A = \frac{1}{2}$.

$4 : \begin{cases} 2 : x''(t) \\ 1 : \text{substitutes } x''(t), x'(t), \text{ and } x(t) \\ \quad \text{into } Ax''(t) + x'(t) + x(t) \\ 1 : \text{answer} \end{cases}$

 4A, 1

**NO CALCULATOR ALLOWED**

**CALCULUS AB**

**SECTION II, Part B**

**Time—45 minutes**

**Number of problems—3**

**No calculator is allowed for these problems.**

Work for problem 4(a)

$x(t) = e^{-t}\sin t$

$v(t) = e^{-t}\cos t + \sin t\, e^{-t} \cdot -1$

$v(t) = e^{-t}(\cos t - \sin t)$

$0 = e^{-t}(\cos t - \sin t)$

$\cos t = \sin t$

$t = \frac{\pi}{4} \quad t = \frac{5\pi}{4}$

| $t$ | $x(t)$ |
|---|---|
| $0$ | $0$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}$ |
| $\frac{5\pi}{4}$ | $-\frac{\sqrt{2}}{2}e^{-\frac{5\pi}{4}}$ |
| $2\pi$ | $0$ |

$t = \frac{5\pi}{4}$. By closed interval test, $\frac{5\pi}{4}$ is the x coordinate of the absolute minimum. value of $x(t)$ on $[0, 2\pi]$

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Work for problem 4(b)

$x(t) = e^{-t}\sin t \qquad x'(t) = e^{-t}\cos t - e^{-t}\sin t$

$x''(t) = -e^{-t}\sin t - \cos t\, e^{-t} - (e^{-t}\cos t - e^{-t}\sin t)$

$x''(t) = \cancel{-e^{-t}\sin t} - e^{-t}\cos t - e^{-t}\cos t \cancel{+ e^{-t}\sin t}$

$x''(t) = -2e^{-t}\cos t$

$Ax''(t) + x'(t) + x(t) = 0$

$A(-2e^{-t}\cos t) + e^{-t}\cos t \cancel{- e^{-t}\sin t + e^{-t}\sin t} = 0$

$-2Ae^{-t}\cos t + e^{-t}\cos t = 0$

$e^{-t}\cos t\,(-2A + 1) = 0$

$A = \frac{-1}{-2}$

$A = \frac{1}{2}$

**NO CALCULATOR ALLOWED**

**CALCULUS AB**

**SECTION II, Part B**

**Time—45 minutes**

**Number of problems—3**

**No calculator is allowed for these problems.**

Work for problem 4(a)

$x'(t) = v(t) = e^{-t}\cos t + \sin t(e^{-t})(-1)$

$= e^{-t}\cos t - e^{-t}\sin t$

$= e^{-t}(\cos t - \sin t)$

$e^{-t}(\cos t - \sin t) = 0$

when $v(t)$ is negative, zero, then positive on the interval $[0, 2\pi]$

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Continue problem 4 on page 11.

Work for problem 4(b)

$v(t) = e^{-t}(\cos t - \sin t)$

$x''(t) = e^{-t}(-\sin t - \cos t) + (\cos t - \sin t)(e^{-t})(-1)$

$e^{-t}(-\sin t - \cos t) - e^{-t}(\cos t - \sin t)$

$e^{-t}(-\cancel{\sin t} - \cos t - \cos t + \cancel{\sin t})$

$e^{-t}(-2\cos t)$

$A e^{-t}(-2\cos t) + e^{-t}(\cos t - \sin t) + e^{-t}\sin t = 0$

$Ae^{-t}(-2\cos t + \cos t - \cancel{\sin t} + \cancel{\sin t}) = 0$

$Ae^{-t}(\cos t) = 0 \quad [0, 2\pi]$

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**NO CALCULATOR ALLOWED**

**CALCULUS AB**

**SECTION II, Part B**

**Time—45 minutes**

**Number of problems—3**

**No calculator is allowed for these problems.**

Work for problem 4(a)

$x'(t)$ or $v(t) = e^{-t}(\cos t) + \sin t(e^{-t} \cdot -1)$

$= e^{-t}\cos t - e^{-t}\sin t$

$= e^{-t}(\cos t - \sin t)$

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Work for problem 4(b)

$x'(t) = e^{-t}(\cos t - \sin t)$

$x''(t) = e^{-t}(-\sin t - \cos t) + (\cos t - \sin t)(e^{-t})(-1)$

$= e^{-t}(-\sin t - \cos t) - e^{-t}(\cos t - \sin t)$

$= e^{-t}(-\cancel{\sin t} - \cos t - \cos t + \cancel{\sin t})$

$= e^{-t}(-2\cos t)$

$= -2e^{-t}\cos t$

$A(e^{-t}\sin t) + (e^{-t}(\cos t - \sin t)) - 2e^{-t}\cos t = 0$

$A(e^{-t}\cancel{\sin t} + \cancel{e^{-t}\cos t} - e^{-t}\cancel{\sin t} - \overset{1}{\cancel{2}}e^{-t}\cos t) = 0$

$A(-1e^{-t}\cos t) = 0$

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## Question 4

### Overview

This problem presented students with a function $x(t)$ describing the position of a particle at time $t$ moving along the $x$-axis over a closed time interval. Part (a) asked for the time, with justification, when the particle was farthest to the left during this time interval. The first derivative of $x(t)$ was required to compute the time and complete the justification. Part (b) required students to substitute the function and the first and second derivatives of $x(t)$ into the equation $Ax''(t) + x'(t) + x(t) = 0$ to find the value of $A$. Students did not have to solve the differential equation to determine the value of $A$.

**Sample: 4A**
**Score: 9**

The student earned all 9 points.

**Sample: 4B**
**Score: 6**

The student earned 6 points: 3 points in part (a) and 3 points in part (b). In part (a) the student earned 2 points for $x'(t)$ and 1 point for setting $x'(t) = 0$. In part (b) the student earned 2 points for $x''(t)$ and 1 point for the correct substitution. The student does not solve for $A$ and thus did not earn the answer point.

**Sample: 4C**
**Score: 4**

The student earned 4 points: 2 points in part (a) and 2 points in part (b). In part (a) the student earned 2 points for $x'(t)$. In part (b) the student earned 2 points for $x''(t)$. The student incorrectly substitutes for $x''(t)$ and does not solve for $A$.