Question 3

Intent of Question

The primary goals of this question are to assess a student's ability to: (1) recognize the random variable of interest, identify its probability distribution, and calculate a probability; (2) use basic probability rules to find a different probability; and (3) use the sampling distribution of the sample mean to identify a characteristic of the manufacturing process that will meet a given specification.

Solution

Part (a):

Let *D* represent the distance a randomly selected ball travels. Since *D* is normally distributed with a mean of 288 yards and a standard deviation of 2.8 yards, we find

$$P(D > 291.2) = P\left(Z > \frac{291.2 - 288}{2.8}\right) > P(Z > 1.14) = 1 - 0.8729 = 0.1271.$$

Part (b):

$$\begin{aligned} P(\text{at least one distance} > 291.2) &= 1 - P(\text{all five distances} \le 291.2) \\ &= 1 - (1 - 0.1271)^5 \\ &= 1 - (0.8729)^5 \\ &= 1 - 0.5068 \\ &= 0.4932 \end{aligned}$$

Part (c):

Since the 99th percentile for a standard normal distribution is 2.33, we can set the appropriate *z*-score equal to 2.33 and solve for the desired mean, say *M*. Thus,

$\frac{291.2 - M}{2.8} = 2.33$ or $M = 291.2 - 2.33 \times 2.8 = 284.676$. In order to be 99 percent certain that a randomly selected ball does not exceed the maximum distance of 291.2 yards, the mean should be set to 284.676 yards.

Scoring

Parts (a), (b), and (c) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Part (a) is essentially correct (E) if the student clearly shows ALL three of the following:

Indicates the distribution is normal.
Specifies BOTH the mean, μ, and the standard deviation, σ.
Calculates the correct probability.

Question 3 (continued)

Part (a) is partially correct (P) if the student:

Calculates the correct probability but fails to specify BOTH μ and σ in the identification of the normal the distribution.

OR

Completely identifies the distribution as normal with BOTH μ and σ correctly specified but fails to calculate the correct probability, e.g., calculates the probability that the ball travels less than 291.2 yards.

OR

Completely identifies the distribution as normal with BOTH μ and σ correctly specified but uses the empirical rule to provide an approximate answer.

Part (a) is incorrect (I) if the student:

Reports a correct probability without showing any work.

OR

Calculates and incorrect probability using an inappropriate distribution.

Notes:

Calculator solution is 0.1265. If this is the only information provided, the response is scored as incorrect (I).

If only the calculator command Normalcdf ($-\infty$, 291.2, 288, 2.8) is provided along with 0.1265, then the response should be scored as partially correct (P).

If the calculator command Normalcdf ($-\infty$, 291.2, 288, 2.8) is provided along with 0.1265 AND the mean and standard deviation are clearly identified, then the response should be scored as essentially correct (E).

If the calculator command Normalcdf ($-\infty$, 291.2, 288, 2.8) AND a shaded/labeled sketch of an appropriate normal distribution are provided along with 0.1265, then the response should be scored as essentially correct (E).

Minor arithmetic or transcription errors will not necessarily lower the score.

Part (b) is essentially correct (E) if the student calculates the correct probability and:

Clearly indicates the distribution is binomial AND specifies both *n* and *p* using the value obtained in part (a)

OR

Correctly applies complement and probability rules using the value obtained in part (a).

Part (b) is partially correct (P) if the student:

Clearly indicates the distribution is binomial AND specifies both *n* and *p* using the value obtained in part (a), but does not calculate the probability correctly.

OR

Calculates the correct probability using the value obtained in part (a) but fails to completely identify the distribution as binomial with both *n* and *p* specified.

OR

Indicates a correct procedure for computing the probability but uses a value of *p* that is different from the value obtained in part (a).

Question 3 (continued)

Part (b) is incorrect (I) if the student

Provides a probability, but no work is shown.

OR

Obtains a probability with an incorrect solution strategy, e.g., $P(\text{at least one distance} > 291.2) = 1 - p^5$ or $P(\text{at least one distance} > 291.2) = 1 - 5p$, where p is the solution to part (a).

Notes:

Calculator solution is 0.4916. If the student uses calculator syntax, BOTH n and p AND the binomial distribution must be identified to be scored essentially correct.

If only the calculator command $1 - \text{binomcdf}(5, 0.1265, 0)$ is provided along with the probability 0.4915, then the response should be scored as partially correct.

Alternative solutions using the binomial distribution with $p = 0.1265$ are:

$$\begin{aligned} P(\text{at least one measurement} > 291.2) &= P(B=1) + P(B=2) + P(B=3) + P(B=4) + P(B=5) \\ &= \binom{5}{1} 0.1265^1 (1-0.1265)^4 + \binom{5}{2} 0.1265^2 (1-0.1265)^3 + \\ &\quad \binom{5}{3} 0.1265^3 (1-0.1265)^2 + \binom{5}{4} 0.1265^4 (1-0.1265)^1 + 0.1265^5 \\ &= 0.368224 + 0.106652 + 0.015445 + 0.001118 + +0.00032 \\ &= 0.491472 \end{aligned}$$

$$\begin{aligned} P(\text{at least one measurement} > 291.2) &= 1 - P(\text{all five distances} > 291.2) \\ &= 1 - \binom{5}{0} 0.1265^0 (1-0.1265)^5 \\ &= 1 - (1-0.1265)^5 \\ &= 1 - (0.8735)^5 \\ &= 1 - 0.5085 \\ &= 0.4915 \end{aligned}$$

Part (c) is essentially correct (E) if the student clearly shows ALL three of the following:

Identifies the 99th percentile for the standard normal distribution

Sets up an appropriate equation

Solves for the desired mean

Part (c) is partially correct (P) if the student:

Recognizes that the 99th percentile for the standard normal distribution must be used and sets up the appropriate equation but does not solve the equation for the desired mean.

Question 3 (continued)

OR

Sets up an appropriate equation using the correct minimum distance but provides an incorrect mean because an incorrect upper tail percentile (say the 99.5^{th} percentile) was used.

OR

Sets up an appropriate equation using the correct percentile of the standard normal distribution but provides an incorrect mean because an incorrect minimum distance was used (say 288 yards) was used.

Part (c) is incorrect (I) if the student

Provides the correct mean, 284.676 yards, but no work is shown

OR

A lower tail percentile is used.

OR

An incorrect mean is calculated with an incorrect solution strategy.

Note: Calculator solution is 284.686 yards.

4 Complete Response

All three parts essentially correct

3 Substantial Response

Two parts essentially correct and one part partially correct

2 Developing Response

Two parts essentially correct and no parts partially correct

OR

One part essentially correct and two parts partially correct

OR

Three parts partially correct

1 Minimal Response

One part essentially correct and either zero or one part partially correct

OR

No parts essentially correct and two parts partially correct

3A1

3. Golf balls must meet a set of five standards in order to be used in professional tournaments. One of these standards is distance traveled. When a ball is hit by a mechanical device, Iron Byron, with a 10-degree angle of launch, a backspin of 42 revolutions per second, and a ball velocity of 235 feet per second, the distance the ball travels may not exceed 291.2 yards. Manufacturers want to develop balls that will travel as close to the 291.2 yards as possible without exceeding that distance. A particular manufacturer has determined that the distances traveled for the balls it produces are normally distributed with a standard deviation of 2.8 yards. This manufacturer has a new process that allows it to set the mean distance the ball will travel.

(a) If the manufacturer sets the mean distance traveled to be equal to 288 yards, what is the probability that a ball that is randomly selected for testing will travel too far?

$\mu = 288$
$\sigma = 2.8$

In order that a ball is tested to travel too far, the ball's traveled distan should exceed 291.2 yards.

Let x = traved distances of the ball

$P(X > 291.2)$ — Because ① distances traveled are normally distributed ② a ball is randomly selected

$= P(z > \frac{x-\mu}{\sigma})$

$= P(z > \frac{291.2-288}{2.8})$

$= P(z > 1.14) = 0.1271$

Therefore, the probability that a ball that is randomly selected for testing will travel too far is 0.1271.

(b) Assume the mean distance traveled is 288 yards and that five balls are independently tested. What is the probability that at least one of the five balls will exceed the maximum distance of 291.2 yards?

In (a), we calculate the probability of a ball to exceed the maximum distance of 291.2 yards is 0.1271. Let $p = 0.1271$

$n = 5$ (five balls)
$p = 0.1271$

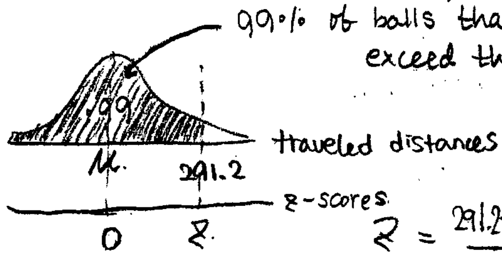
This is binomial distribution, because five balls are independently tested, and "p" remains the same.

$1 - \underbrace{{}_5C_5(1-0.1271)^5}_{\text{probability that none of five balls exceed 291.2 yards}} = 0.493$

Therefore, the probability that at least one of the five balls will exceed the maximum distance of 291.2 yards is 0.493.

(c) If the manufacturer wants to be 99 percent certain that a randomly selected ball will not exceed the maximum distance of 291.2 yards, what is the largest mean that can be used in the manufacturing process?

99% of balls that do not exceed the 291. yards.



$z = \frac{291.2-\mu}{2.8} = 2.33$

$\therefore \mu = 284.676$

Therefore, the largest mean that can be used in this manufactu process is 284.676 yards.

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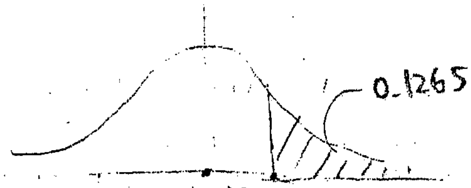
3B1

3. Golf balls must meet a set of five standards in order to be used in professional tournaments. One of these standards is distance traveled. When a ball is hit by a mechanical device, Iron Byron, with a 10-degree angle of launch, a backspin of 42 revolutions per second, and a ball velocity of 235 feet per second, the distance the ball travels may not exceed 291.2 yards. Manufacturers want to develop balls that will travel as close to the 291.2 yards as possible without exceeding that distance. A particular manufacturer has determined that the distances traveled for the balls it produces are normally distributed with a standard deviation of 2.8 yards. This manufacturer has a new process that allows it to set the mean distance the ball will travel.

(a) If the manufacturer sets the mean distance traveled to be equal to 288 yards, what is the probability that a ball that is randomly selected for testing will travel too far?

Because the distances traveled for the balls produced are normally distributed, Normal approximation can be used.



μ = 288 yards, 291.2 yards

σ = 2.8 yards. z-score = $\frac{291.2-288}{2.8} = 1.143$

$\therefore$ 12.65 %

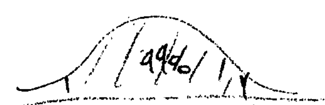
(b) Assume the mean distance traveled is 288 yards and that five balls are independently tested. What is the probability that at least one of the five balls will exceed the maximum distance of 291.2 yards?

This probability = 1 − the probability that no balls exceed 291.2 yards

$= 1 - \binom{5}{0}(0.1265)^0(0.8735)^5$

$\doteq 0.4915$

(c) If the manufacturer wants to be 99 percent certain that a randomly selected ball will not exceed the maximum distance of 291.2 yards, what is the largest mean that can be used in the manufacturing process?



z-score $\doteq$ 2.576

The largest value of 99% interval.

$\mu + 2.576 \times 2.8 \leq 291.2$

$\therefore \mu \leq 283.987$

$\therefore$ The largest mean = 283.987 for the purpose

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3. Golf balls must meet a set of five standards in order to be used in professional tournaments. One of these standards is distance traveled. When a ball is hit by a mechanical device, Iron Byron, with a 10-degree angle of launch, a backspin of 42 revolutions per second, and a ball velocity of 235 feet per second, the distance the ball travels may not exceed 291.2 yards. Manufacturers want to develop balls that will travel as close to the 291.2 yards as possible without exceeding that distance. A particular manufacturer has determined that the distances traveled for the balls it produces are normally distributed with a standard deviation of 2.8 yards. This manufacturer has a new process that allows it to set the mean distance the ball will travel.

(a) If the manufacturer sets the mean distance traveled to be equal to 288 yards, what is the probability that a ball that is randomly selected for testing will travel too far?

$P(X = \cancel{288}\ 291.2) = P(z = \frac{-288 + 291.2}{2.8}) = P(z = +1.14) = 0.208$ (20.8%)

(b) Assume the mean distance traveled is 288 yards and that five balls are independently tested. What is the probability that at least one of the five balls will exceed the maximum distance of 291.2 yards?

P(at least one will exceed 291.2 yards) = 1 − P(none will exceed 291.2).

$= 1 - P^5(X < 291.2) = 1 - P^5(z < 1.14) = 1 - 0.87^5 = 0.502$ (50.2%)

(c) If the manufacturer wants to be 99 percent certain that a randomly selected ball will not exceed the maximum distance of 291.2 yards, what is the largest mean that can be used in the manufacturing process?

$P(X < 291.2) = P(z < \frac{291.2 - \mu_0}{2.8})$

$\frac{291.2 - \mu_0}{2.8} = 2.58$

$\mu_0 = 283.976$

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Question 3

Sample: 3A
Score: 4

In part (a) the essay recognizes that the random variable of interest is the distance that the golf ball travels; it clearly indicates the mean and standard deviation of the distribution and correctly computes the probability that a golf ball will travel more than 291.2 yards using the appropriate normal distribution. The solution is supported by displaying the steps in the probability calculations and providing a sketch of the standard normal density function with a shaded area corresponding to the appropriate probability. The binomial distribution and the complement rule are correctly applied in part (b) to obtain the probability that at least one of the five tested balls will travel farther than 291.2 yards. The 99th percentile of the standard normal distribution is correctly identified in part (c) and used to compute the desired mean of 284.676 yards. A sketch of the standard normal distribution is used to organize thinking and illustrate the link between the probability that a ball travels less than 291.2 yards and the mean and standard deviation of the desired normal distribution of distances.

Sample: 3B
Score: 3

In part (a) the essay recognizes that the random variable of interest is the distance that the golf ball travels; it clearly indicates the mean and standard deviation of the distribution and correctly computes the probability that a golf ball will travel more than 291.2 yards using the appropriate normal distribution. The steps in the probability calculation are not displayed, but the solution is supported by a sketch of a normal density function with a shaded area corresponding to the appropriate probability. The binomial distribution and the complement rule are correctly applied in part (b) to obtain the probability that at least one of the five tested balls will travel farther than 291.2 yards. The response to part (c) uses the 99.5th percentile of the normal distribution instead of the appropriate 99th percentile and obtains a mean that is too small. This error was not uncommon.

Sample: 3C
Score: 2

In part (c) the correct probability formula is presented, but the essay uses the 99.5th percentile of the normal distribution instead of the appropriate 99th percentile and obtains a mean that is too small. The response to part (a) also sets up the correct probability formula but fails to obtain the correct probability. The solution to part (b) makes appropriate use of probability rules, but the value of the probability that the ball will travel less than 291.2 yards, 0.87, is inconsistent with the solution to part (a), and no information is provided about how 0.87 is selected. This response shows some understanding of use of probability rules and the binomial and normal distribution in each part, but it fails to correctly apply that knowledge to obtain correct solutions.