The function $f$ is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers $x$ for which the series converges. The function $g$ is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers $x$ for which the series converges.

(a) Find the interval of convergence of the power series for $f$. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether $y$ has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

(a) The series converges when $1 - x < 1$.

When $x = 1$, the series is

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

(b) $f'(x) = -\frac{1}{2} + \frac{4}{3} x - \frac{9}{4} x^2 + \cdots$ and $f'(0) = -\frac{1}{2}$.

$g'(x) = -\frac{1}{2!} + \frac{2}{4!} x - \frac{3}{6!} x^2 + \cdots$ and $g'(0) = -\frac{1}{2}$.

$y'(0) = f'(0) - g'(0) = 0$

$f''(0) = \frac{4}{3}$ and $g''(0) = \frac{2}{4!} = \frac{1}{12}$.

Thus, $y''(0) = \frac{4}{3} - \frac{1}{12} > 0$.

Since $y'(0) = 0$ and $y''(0) > 0$, $y$ has a relative minimum at $x = 0$. 
Work for problem 6(a)

\[
\lim_{n \to \infty} \frac{(-1)^{n+1} (n+1) x^{n+1}}{n+2} = \lim_{n \to \infty} \frac{(-1)^{n+1} (n+1) x^{n} x}{n+1} = \lim_{n \to \infty} \frac{(-1)^{n+1} (n+1) x^{n}}{n+1}
\]

\[
-5 \leq R = 1
\]

diverges due to the

\[
\lim_{n \to \infty} (-1)^{n+1} \frac{n}{n+1} = 1
\]


diverges due to the

\[
\lim_{n \to \infty} (-1)^{n} n = 0
\]

\[
\lim_{n \to \infty} \frac{(-1)^{n} n}{n+1} = 0
\]

\[
\lim_{n \to \infty} \frac{(-1)^{n+1} x^{n+1}}{n+2} = 0
\]

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Continue problem 6 on page 15.
Work for problem 6(b)

\[ g(x) = \cos x \]

\[ f'(x) = -\frac{1}{2}x + \frac{1}{3}x^\frac{1}{3} \]

\[ f'(0) = -\frac{1}{2} \]

\[ g'(x) = -\frac{1}{2} + \frac{2}{4}x \]

\[ g'(0) = -\frac{1}{2} \]

\[ f''(x) = \frac{1}{3} \]

\[ g''(x) = \frac{1}{12} \]

\[ g''(0) = \frac{1}{12} - \frac{1}{12} = \frac{1}{12} \]

has a relative minimum at \( x = 0 \) because the derivative shows there is a critical point at \( x = 0 \) and the 2nd derivative shows a positive concavity, meaning the function values are decreasing up till \( x = 0 \) and increasing after, therefore showing a relative minimum.

END OF EXAM

THE FOLLOWING INSTRUCTIONS APPLY TO THE COVERS OF THE SECTION II BOOKLET.

- MAKE SURE YOU HAVE COMPLETED THE IDENTIFICATION INFORMATION AS REQUESTED ON THE FRONT AND BACK COVERS OF THE SECTION II BOOKLET.
- CHECK TO SEE THAT YOUR AP NUMBER LABEL APPEARS IN THE BOX(ES) ON THE COVER(S).
- MAKE SURE YOU HAVE USED THE SAME SET OF AP NUMBER LABELS ON ALL AP EXAMS YOU HAVE TAKEN THIS YEAR.
Work for problem 6(a)

\[ \frac{(n+1)x^{n+1}}{(n+1)+1} \cdot \frac{n+1}{nx^n} = \frac{(n^2+2n+1)x}{(n^2+n)} = 3x \]

\[ -1 \leq 3x \leq 1 \]

\[ -\frac{1}{3} \leq x \leq \frac{1}{3} \]

\[ \frac{(-1)^n n \left(\frac{x}{3}\right)^n}{n+1} \rightarrow \text{diverges} \]

\[ \frac{(-1)^n n \left(\frac{x}{3}\right)^n}{n+1} \rightarrow \text{converges} \]

* When \(-\frac{1}{3} \leq x \leq \frac{1}{3}\), the power series for \( f \) converges.

Continue problem 6 on page 15.
Work for problem 6(b)

\[ y' = (x) - g'(x) \]
\[ y'(0) = \frac{1}{2} + \frac{1}{2} = \frac{1}{2} \]
\[ y'' = f''(x) - g''(x) \]
\[ y''(0) = \frac{1}{3} - \frac{1}{12} = \frac{1}{12} \]

\[ f'(x) = -\frac{1}{2} + \frac{4}{3} \cdots \]
\[ f''(x) = \frac{4}{3} \]
\[ g'(x) = -\frac{1}{2} + \frac{1}{12} x \]
\[ g''(x) = \frac{1}{12} \]

\( y \) has a relative minimum at \( x = 0 \) because the derivative of \( y \) at \( x = 0 \) is 0 and the second derivative of \( y \) at \( x = 0 \) is > 0, meaning the graph of \( y \) is concave up at this point.

STOP
END OF EXAM

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Work for problem 6(a)

\[ f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n + 1} \]

\[ L = \lim_{n \to \infty} \left| \frac{a(n+1)}{a_n} \right| \]

\[ L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (n+1) x^{n+1}}{n + d} \cdot \frac{n + 1}{(-1)^{n+1} n x^5} \right| \]

\[ L = \lim_{n \to \infty} \left| (-1) x \cdot \frac{(n+1)^2}{(n+d)(n)} \right| \]

\[ L = x \]

\[ x < 1 \]

\[ x < 1 \]
Work for problem 6(b)  

\[ y = f(x) - g(x) \]

\[ y'(x) = \left[ -\frac{1}{2} + \frac{4x}{3} + \ldots \right] - \left[ -\frac{1}{2} + \frac{2y}{4!} + \ldots \right] \]

\[ y'(0) = -\frac{1}{2} - (\frac{1}{2}) \]

\[ y'(0) = 0 \]

\[ y''(x) = \left[ \frac{4}{3} - \frac{18}{4} x + \ldots \right] - \left[ -\frac{2}{24} - \frac{6x}{6!} + \ldots \right] \]

\[ y''(0) = \frac{4}{3} - \frac{1}{12} = \frac{15}{12} \]

\[ x < 0 \]

Relative minimum

STOP

END OF EXAM

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Question 6

Overview

This problem dealt with power series. Students were given the power series expansions of two functions, \( f \) and \( g \). In part (a) they were asked to find the interval of convergence of the power series for \( f \). Part (b) dealt with the graph of \( y = f(x) - g(x) \). Students had to know how to read or compute the values of the first and second derivatives of \( y \) at \( x = 0 \) from the series for \( f \) and \( g \). They then needed to use this information to describe the nature of the critical point of \( y \) at \( x = 0 \).

Sample: 6A
Score: 9

The student earned all 9 points. In part (b) the student restarts the problem on the third line and earned all points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a) and 4 points in part (b). In part (a) the student sets up an incorrect Ratio Test and does not compute a limit. The student earned the 2 points by stating the correct interval of convergence for the ratio and testing the endpoints. In part (b) the student calculates \( y'(0) \) and \( y''(0) \) and arrives at the correct conclusion with good reasoning.

Sample: 6C
Score: 4

The student earned 4 points: 1 point in part (a) and 3 points in part (b). In part (a) the student sets up the ratio correctly but does not identify the correct interval of convergence. In part (b) the student correctly calculates \( y'(0) \) and \( y''(0) \) and arrives at a correct conclusion. However, the student does not include a reason for the answer.