Question 5

Let f be a function with $f(4) = 1$ such that all points (x, y) on the graph of f satisfy the differential equation

$$\frac{dy}{dx} = 2y(3 - x).$$

Let g be a function with $g(4) = 1$ such that all points (x, y) on the graph of g satisfy the logistic differential equation

$$\frac{dy}{dx} = 2y(3 - y).$$

(a) Find $y = f(x)$.

(b) Given that $g(4) = 1$, find $\lim\limits_{x\to\infty} g(x)$ and $\lim\limits_{x\to\infty} g'(x)$. (It is not necessary to solve for $g(x)$ or to show how you arrived at your answers.)

(c) For what value of y does the graph of g have a point of inflection? Find the slope of the graph of g at the point of inflection. (It is not necessary to solve for $g(x)$.)

(a) $\frac{dy}{dx} = 2y(3 - x)$

$\frac{1}{y}\,dy = 2(3 - x)\,dx$

$\ln|y| = 6x - x^2 + C$

$0 = 24 - 16 + C$

$C = -8$

$\ln|y| = 6x - x^2 - 8$

$y = e^{6x - x^2 - 8}$ for $-\infty < x < \infty$

$$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solution} \end{cases}$$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(b) $\lim\limits_{x\to\infty} g(x) = 3$

$\lim\limits_{x\to\infty} g'(x) = 0$

$$2 : \begin{cases} 1 : \lim\limits_{x\to\infty} g(x) = 3 \\ 1 : \lim\limits_{x\to\infty} g'(x) = 0 \end{cases}$$

(c) $\frac{d^2y}{dx^2} = (6 - 4y)\frac{dy}{dx}$

Because $\frac{dy}{dx} \neq 0$ at any point on the graph of g, the concavity only changes sign at $y = \frac{3}{2}$, half the carrying capacity.

$\left.\frac{dy}{dx}\right|_{y=3/2} = 2\left(\frac{3}{2}\right)\left(3 - \frac{3}{2}\right) = \frac{9}{2}$

$$2 : \begin{cases} 1 : y = \frac{3}{2} \\ 1 : \left.\frac{dy}{dx}\right|_{y=3/2} \end{cases}$$

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Work for problem 5(a)

$\frac{dy}{dx} = 2y(3-x)$

$dy = 2y(3-x)dx$

$\int \frac{1}{2y} dy = \int (3-x)dx$

$\frac{1}{2} \ln|y| = 3x - \frac{x^2}{2} + C$

$\ln|y| = 2(3x - \frac{x^2}{2} + C)$

$\ln|y| = 6x - x^2 + C$

$y = e^{6x-x^2+C}$

$y = Ce^{6x-x^2}$

$f(4) = 1 = Ce^{24-16} = Ce^8$

$\frac{1}{e^8} = \frac{C\cancel{e^8}}{\cancel{e^8}}$

$\boxed{y = f(x) = \frac{1}{e^8} e^{6x-x^2}}$

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NO CALCULATOR ALLOWED

Work for problem 5(b)

$\frac{dP}{dt} = ky(L-y)$

L represents carrying capacity

$\frac{dy}{dx} = 2y(3-y)$

3 is the carrying capacity

$\lim_{x\to\infty} g(x) = $ 3

$\lim_{x\to\infty} g'(x) = $ 0

Work for problem 5(c)

$\lim_{x\to\infty} g(x) = 3$

$\frac{dy}{dx} = 2y(3-y)$

This is a logistic function with carrying capacity of 3

There is always a point of inflection in logistic functions whenever y is half of the carrying capacity

∴ $\boxed{y=1.5 \text{ is the inflection point}}$

$\left.\frac{dy}{dx}\right|_{y=1.5} = 2(1.5)(3-1.5)$

$= 3(1.5) = \boxed{4.5}$

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 5B

NO CALCULATOR ALLOWED

Work for problem 5(a)

$\frac{dy}{dx} = 2y(3-x)$

$\frac{dy}{2y} = (3-x)\,dx$

$\int \frac{dy}{2y} = \int (3-x)\,dx$

$\frac{1}{2}\ln|y| = 3x - \frac{x^2}{2} + c$

$\ln|y| = 6x - x^2 + c$

$y = e^{6x-x^2+c}$ $(e^c = C)$

$= Ce^{6x-x^2}$

$f(4) = 1$

$1 = Ce^8$

$\therefore C = \frac{1}{e^8} = e^{-8}$

$\therefore y = e^{-8} \cdot e^{6x-x^2}$

$= e^{6x-x^2-8}$

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NO CALCULATOR ALLOWED

Work for problem 5(b)

$$\frac{dy}{2y(3-y)} = dx$$

$$\frac{dy}{6y} + \int \frac{dy}{3(6-2y)} = \int dx$$

$$\frac{1}{6}\ln|y| + -\frac{1}{6}\ln|6-2y| = x + C$$

$$\ln y - \ln|6-2y| = 6x + C$$

$$\ln\left|\frac{y}{6-2y}\right| = 6x + c$$

$$\ln\left|\frac{6-2y}{y}\right| = -6x + c$$

$$\frac{6-2y}{y} = e^{-6x+c}$$

$$= Ce^{-6x} \quad (C = \pm e^{c})$$

$\lim_{x\to\infty} g(x) =$ 4

$\lim_{x\to\infty} g'(x) =$ 0

$\boxed{g(4)=1} \rightarrow 4 = Ce^{-24}$ — $g(x) = 4e^{24}e^{-24}$ $= 4$

Work for problem 5(c)

point of inflection $= g''(x) = 0.$ ~~and change~~ (change its sign)

$$\left(\frac{dy}{dx}\right)' = (6y - 2y^2)'$$

$$= 6\frac{dy}{dx} - 4y\frac{dy}{dx}$$

$$= 6(6y-2y^2) - 4y(6y-2y^2)$$

$$= 36y - 12y^2 - 24y^2 + 8y^3$$

$$= 8y^3 - 36y^2 + 36y = 0$$

$$y = 0, \frac{3}{2}, 3$$

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5C

NO CALCULATOR ALLOWED

Work for problem 5(a)

$$\frac{dy}{dx} = 2y(3-x)$$

$$\int dy = \int [2y(3-x)]\,dx$$

$$y = \int [6y - 2xy]\,dx$$

$$y = f(x) = 6xy - x^2y + C$$

$$f(4) = 6 \times 4 \times 1 - 4^2 \times 1 + C$$

$$= 24 - 16 + C = 1$$

$$C = -8$$

$$\therefore \; f(x) = 6xy - x^2y - 8$$

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5C

NO CALCULATOR ALLOWED

Work for problem 5(b)

$\lim_{x \to \infty} g(x) =$ $\frac{1}{3}$

$\lim_{x \to \infty} g'(x) =$ 0

Work for problem 5(c)

In the point of inflection, $\frac{d^2y}{dx^2} = 0$ and the concavity of $g(x)$ changes.

(i) $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(6y - 2y^2)$

$= 6 \times \frac{dy}{dx} - 4y \times \frac{dy}{dx}$

$= (6-4y) \times \frac{dy}{dx}$

$= (6-4y)(2y(3-y)) = 0$

$\therefore y = \frac{3}{2}$ ($y \neq 3$ and $y \neq 0$, since the concavity of $g(x)$ does not change when $\frac{dy}{dx} = 0$)

(ii) slope of graph g at $y = 3/2$.

: $\frac{dy}{dx} = 2y(3-y) = 2 \times 3/2 \left(3 - \frac{3}{2}\right) = 3 \times \frac{3}{2} = \frac{9}{2}$

Answer: $y = \frac{3}{2}$; slope of graph g at the point of inflection is $\frac{9}{2}$

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Question 5

Overview

This problem asked students to work with two differential equations. In part (a) students had to find the particular solution $f(x)$ to a separable differential equation satisfying a given initial condition. Parts (b) and (c) tested students' knowledge of the behavior of a solution $g(x)$ to a logistic differential equation that was superficially similar to, but in fact quite different from, the differential equation solved in part (a). It was not necessary for students to solve for this particular solution. Part (b) tested their knowledge of the limiting behavior of a logistic equation: if the initial y value is positive, then $g(x)$ will approach the carrying capacity, which is the positive root of the quadratic polynomial in y, and $g'(x)$ will approach 0. Even without specific knowledge of the actual solution, students should have been able to determine this information directly from the logistic differential equation. Part (c) tested student recognition that for this particular solution, the point of inflection for the graph of g will occur where $g'(x)$ is greatest. This occurs at the maximum value of the quadratic polynomial in y, or halfway between the two roots (half the carrying capacity). The graph of this particular solution will have a point of inflection since the initial y value is less than half the carrying capacity. The slope at this point could be computed directly from the differential equation.

Sample: 5A
Score: 9

The student earned all 9 points. In parts (b) and (c) the student uses knowledge of the general properties of the solution to a logistic differential equation to help in answering questions about the limits and the point of inflection.

Sample: 5B
Score: 6

The student earned 6 points: all 5 points in part (a) and 1 point in part (b). In part (b) the student attempts to solve the logistic differential equation using the technique of partial fractions. This is not necessary, and that work was not graded. The student does write the correct limit of the derivative and earned 1 point. In part (c) the student uses implicit differentiation and the chain rule to compute the second derivative. The student reports all three zeros as the answer, however, and thus did not earn the first point. The student could still have earned both points in part (c) by identifying $y = \frac{3}{2}$ as the only value of y where the graph of g has an inflection point and computing the slope at that value.

Sample: 5C
Score: 3

The student earned 3 points: 1 point in part (b) and 2 points in part (c). In part (a) the student does not separate the variables. The student attempts to antidifferentiate an expression that still contains both the y and x variables and therefore received none of the 5 available points. In part (b) the student has the correct limit for the derivative, but the answer for the limit of the function is the reciprocal of the carrying capacity. The work in part (c) earned both points.

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