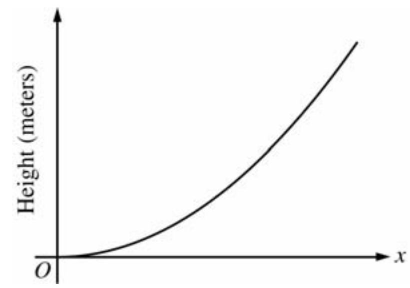
Question 3



The figure above is the graph of a function of x, which models the height of a skateboard ramp. The function meets the following requirements.

(i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.
(ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.
(iii) Between $x = 0$ and $x = 4$, the function is increasing.

(a) Let $f(x) = ax^2$, where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.

(b) Let $g(x) = cx^3 - \dfrac{x^2}{16}$, where c is a nonzero constant. Find the value of c so that g meets requirement (ii) above. Show the work that leads to your answer.

(c) Using the function g and your value of c from part (b), show that g does not meet requirement (iii) above.

(d) Let $h(x) = \dfrac{x^n}{k}$, where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.

(a) $f(4) = 1$ implies that $a = \dfrac{1}{16}$ and $f'(4) = 2a(4) = 1$ implies that $a = \dfrac{1}{8}$. Thus, f cannot satisfy (ii).

$2 : \begin{cases} 1 : a = \dfrac{1}{16} \text{ or } a = \dfrac{1}{8} \\ 1 : \text{shows } a \text{ does not work} \end{cases}$

(b) $g(4) = 64c - 1 = 1$ implies that $c = \dfrac{1}{32}$.

When $c = \dfrac{1}{32}$, $g'(4) = 3c(4)^2 - \dfrac{2(4)}{16} = 3\left(\dfrac{1}{32}\right)(16) - \dfrac{1}{2} = 1$

1 : value of c

(c) $g'(x) = \dfrac{3}{32}x^2 - \dfrac{x}{8} = \dfrac{1}{32}x(3x - 4)$

$g'(x) < 0$ for $0 < x < \dfrac{4}{3}$, so g does not satisfy (iii).

$2 : \begin{cases} 1 : g'(x) \\ 1 : \text{explanation} \end{cases}$

(d) $h(4) = \dfrac{4^n}{k} = 1$ implies that $4^n = k$.

$h'(4) = \dfrac{n4^{n-1}}{k} = \dfrac{n4^{n-1}}{4^n} = \dfrac{n}{4} = 1$ gives $n = 4$ and $k = 4^4 = 256$.

$h(x) = \dfrac{x^4}{256} \Rightarrow h(0) = 0.$

$h'(x) = \dfrac{4x^3}{256} \Rightarrow h'(0) = 0$ and $h'(x) > 0$ for $0 < x < 4$.

$4 : \begin{cases} 1 : \dfrac{4^n}{k} = 1 \\ 1 : \dfrac{n4^{n-1}}{k} = 1 \\ 1 : \text{values for } k \text{ and } n \\ 1 : \text{verifications} \end{cases}$

Work for problem 3(a)

according to (ii), $f(4)=1$, $f'(4)=1$

$f(x)=ax^2 \rightarrow 16a=1 \quad a=\frac{1}{16}$

$f'(x)=2ax \rightarrow 8x=1 \quad a=\frac{1}{8}$

$\frac{1}{16} \neq \frac{1}{8}$

$\therefore$ it's impossible to find a value for a so that f meets requirement (ii).

Work for problem 3(b)

according to (ii), $g(4)=1$, $g'(4)=1$

$g(x)=cx^3-\frac{x^2}{16} \rightarrow 64c-\frac{16}{16}=64c-1=1 \qquad c=\frac{1}{32}$

$g'(x)=3cx^2-\frac{1}{8}x \rightarrow 3\cdot16\cdot c-\frac{1}{2}=48c-\frac{1}{2}=1 \qquad c=\frac{1}{32}$

$\therefore c=\frac{1}{32}$

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Continue problem 3 on page 9

Work for problem 3(c)

$g'(x) = \frac{3}{32}x^2 - \frac{1}{8}x = \frac{3}{32}x\left(x-\frac{4}{3}\right)$

$\therefore$ $x<0$: $g'(x)>0$, $g(x)$ increasing

$0<x<\frac{4}{3}$: $g'(x)<0$, $g(x)$ decreasing

$\frac{4}{3}<x$: $g'(x)>0$, $g(x)$ increasing

$\because$ $g(x)$ do not increase when $0<x<\frac{4}{3}$. So it does not meet requirement (iii)

Work for problem 3(d)

according to (ii), $h(4)=1$, $h'(4)=1$

$h(x) = \frac{x^n}{k} \rightarrow \frac{4^n}{k}=1$

$h'(x) = \frac{n}{k}\cdot x^{n-1} \rightarrow \frac{n}{k}\cdot 4^{n-1}=1$

$\therefore$ $4^n=k$, $4^{n-1}\cdot n=k$

$\therefore$ $n=4$ $k=256$

$\therefore h(x) = \frac{x^4}{256}$

$h(0)=0$, $h'(0)=0$ $\rightarrow$ meet requirement (i)

$h'(x) = \frac{4}{256}\cdot x^3 = \frac{1}{64}x^3$ $x>0$, $h'(x)>0$ $\therefore h(x)$ increasing $\rightarrow$ meet requirement (iii).

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Work for problem 3(a)

$f(x) = ax^2$

$f'(x) = 2ax$

to satisfy (ii), $x = 4$

$f(4) = 1$

$f'(4) = 1$

$f(x) = 16a = 1$

$a = \frac{1}{16}$

$f'(x) = 2 \cdot 4a = 1$

$a = \frac{1}{8}$

- There is no value a that satisfies requirement (ii)

Work for problem 3(b)

$g(x) = cx^3 - \frac{x^2}{16}$

$g'(x) = 3cx^2 - \frac{x}{8}$

$g(4) = 64c - 1 = 1$ $c = \frac{1}{32}$

$g'(4) = 48c - .5 = 1$ $c = \frac{1}{32}$

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Work for problem 3(c)

$g(x) = \frac{1}{32}x^3 - \frac{x^2}{16}$ $g'(0) = 0$

$g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = 0$ $g'(4) = 1$

$x(\frac{3}{32}x - \frac{1}{8}) = 0$

$x=0$ $x = 1.333$

Because $g'(0) = 0$, $g(x)$ is not increasing at $x=0$, thus it does not satisfy requirement (iii)

Work for problem 3(d)

$h(x) = \frac{x^n}{k}$ $\frac{4^n}{k} = 1$ $4^n = k$

$h'(x) = \frac{nx^{n-1}}{k}$ $\frac{n4^{n-1}}{k} = 1$ $n4^{n-1} = k$

$4^n = n4^{n-1}$

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Work for problem 3(a)

$f(x) = ax^2$

$y = ax^2$

$1 = 16a$

$a = \frac{1}{16}$

$y = \frac{1}{16}x^2$

$a = \frac{1}{16}$

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Work for problem 3(b)

$x = 4$
$y = 1$

$1 = 64c - 1$

$2 = 64c$

$c = \frac{1}{32}$

Continue problem 3 on page 9

3C

Work for problem 3(c)

$$g(x) = \frac{x^3}{32} - \frac{x^2}{16}$$

$$= \frac{x^3 - 2x^2}{32}$$

$x = 0$
$y = 0$

$x = 1$
$y = -\frac{1}{32}$

$x = 2$
$y = 0$

$x = 3$
$y = 0$

$x = 4$
$y = 1$

Work for problem 3(d)

$$h(x) = \frac{x^n}{k}$$

$$1 = \frac{4^n}{k}$$

$$k = 4^n.$$

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END OF PART A OF SECTION II

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Question 3

Overview

This problem presented three requirements that had to be satisfied by the graph of a function modeling the height of a skateboard ramp. Students were asked to investigate three families of functions that might be used for such a model. In part (a) they were asked to show that no quadratic of the form ax^2 would satisfy the second requirement. In part (b) they were asked to find the coefficient c for which the cubic $cx^3 - \frac{x^2}{16}$ would meet the second requirement, but then show in part (c) that the cubic with this value of c does not meet the third requirement. Finally, in part (d) students were asked to find the values of n and k for which the power function $\frac{x^n}{k}$ would meet all three requirements.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). The student's work is correct in parts (a) and (b). In part (c) the student earned 1 point for finding the derivative of g. The student does not explain why g is not increasing between $x = 0$ and $x = 4$ and so did not earn the second point in this part. In part (d) the student sets up correct equations to find n and k, earning 1 point for each equation, but does not find n or k and thus cannot show that the function h meets requirements (i) and (iii).

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (d). In part (a) the student finds the value of a for which $f(4) = 1$, which earned the first point, but fails to show that this value of a does not work to meet requirement (ii). In part (b) the student uses the information about g to find the desired value of c. In part (c) the student's calculations of the values of the function g at integer values of x earned no points (and the value at $x = 3$ is incorrect). However, both points could have been earned in part (c) with those calculations if the student had gone on to observe that the value of y at $x = 1$ is less than the value of y at $x = 0$, and hence the function g is not increasing on the interval $0 \le x \le 4$. In part (d) the student earned 1 point for using the information about $h(4)$ to write an equation for n and k but has no other work.