The figure above is the graph of a function of \( x \), which models the height of a skateboard ramp. The function meets the following requirements.

(i) At \( x = 0 \), the value of the function is 0, and the slope of the graph of the function is 0.
(ii) At \( x = 4 \), the value of the function is 1, and the slope of the graph of the function is 1.
(iii) Between \( x = 0 \) and \( x = 4 \), the function is increasing.

(a) Let \( f(x) = ax^2 \), where \( a \) is a nonzero constant. Show that it is not possible to find a value for \( a \) so that \( f \) meets requirement (ii) above.

\[(a) \quad f(4) = 1 \text{ implies that } a = \frac{1}{16} \text{ and } f'(4) = 2a(4) = 1\]

implies that \( a = \frac{1}{8} \). Thus, \( f \) cannot satisfy (ii).

(b) Let \( g(x) = cx^3 - \frac{x^2}{16} \), where \( c \) is a nonzero constant. Find the value of \( c \) so that \( g \) meets requirement (ii) above. Show the work that leads to your answer.

\[(b) \quad g(4) = 64c - 1 = 1 \text{ implies that } c = \frac{1}{32}.
When } c = \frac{1}{32}, \quad g'(4) = 3c(4)^2 - \frac{2(4)}{16} = 3\left(\frac{1}{32}\right)(16) - \frac{1}{2} = 1\]

(c) \( g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = \frac{1}{32}x(3x - 4) \)
\( g'(x) < 0 \text{ for } 0 < x < \frac{4}{3}, \text{ so } g \text{ does not satisfy (iii).} \)

(d) Let \( h(x) = \frac{x^n}{k} \), where \( k \) is a nonzero constant and \( n \) is a positive integer. Find the values of \( k \) and \( n \) so that \( h \) meets requirement (ii) above. Show that \( h \) also meets requirements (i) and (iii) above.

\[(d) \quad h(4) = \frac{4^n}{k} = 1 \text{ implies that } 4^n = k.
\]
\[h'(4) = \frac{n4^{n-1}}{k} = \frac{n4^{n-1}}{4^n} = \frac{n}{4} = 1 \text{ gives } n = 4 \text{ and } k = 4^4 = 256.\]
\[h(x) = \frac{x^4}{256} \Rightarrow h(0) = 0.\]
\[h'(x) = \frac{4x^3}{256} \Rightarrow h'(0) = 0 \text{ and } h'(x) > 0 \text{ for } 0 < x < 4.\]
Work for problem 3(a)

According to (ii), \( f(4) = 1, \quad f'(4) = 1 \)

\[
f(x) = ax^2 \quad \rightarrow \quad 16a = 1 \quad \Rightarrow \quad a = \frac{1}{16}
\]

\[
f'(x) = 2ax \quad \rightarrow \quad 8a = 1 \quad \Rightarrow \quad a = \frac{1}{8}
\]

\[
\frac{1}{16} \neq \frac{1}{8}
\]

\[
\therefore \text{it's impossible to find a value for } a \text{ so that } f \text{ meets requirement (ii)}.
\]

Work for problem 3(b)

According to (ii), \( g(4) = 1, \quad g'(4) = 1 \)

\[
g(x) = cx^3 - \frac{x^2}{16} \quad \rightarrow \quad 64c - \frac{16}{16} = 64c - 1 = 1 \quad \Rightarrow \quad c = \frac{1}{32}
\]

\[
g'(x) = 3cx^2 - \frac{1}{8}x \quad \rightarrow \quad 3 \cdot 16c - \frac{1}{8} = 48c - \frac{1}{8} = 1 \quad \Rightarrow \quad c = \frac{1}{32}
\]

\[
\therefore \quad c = \frac{1}{32}
\]
Work for problem 3(c)

\[ g'(x) = \frac{3}{2} x^2 - \frac{1}{8} x = \frac{3}{32} x (x - \frac{4}{3}) \]

\[ \begin{align*} 
& \text{if } x < 0 : \; g'(x) > 0 \; , \; g(x) \text{ increasing} \\
& 0 < x < \frac{4}{3} : \; g'(x) < 0 \; , \; g(x) \text{ decreasing} \\
& \frac{4}{3} < x : \; g'(x) > 0 \; , \; g(x) \text{ increasing} \\
\end{align*} \]

\[ g(x) \text{ do not increase when } 0 < x < \frac{4}{3}. \; \text{So it does not meet requirement (iii)} \]

Work for problem 3(d)

According to (ii), \[ h(4) = 1, \; h'(4) = 1 \]

\[ h(x) = \frac{x^n}{e} \quad \Rightarrow \quad \frac{4^n}{e} = 1 \]

\[ h'(x) = \frac{n}{e} x^{n-1} \quad \Rightarrow \quad \frac{n}{e} \cdot 4^{n-1} = 1 \]

\[ 4^n = e, \quad 4^{n-1} \cdot n = e \]

\[ n = 4, \quad e = 256 \]

\[ h(x) = \frac{x^4}{256} \]

\[ h(0) = 0, \; h'(0) = 0 \quad \Rightarrow \text{meet requirement (i)} \]

\[ h'(x) = \frac{4}{256} \cdot x^3 = \frac{1}{64} x^3 \quad x > 0, \; h'(x) > 0 \quad \Rightarrow \text{h(x) increasing} \quad \Rightarrow \text{meet requirement (ii).} \]

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.
Work for problem 3(a)

\[ f(x) = ax^2 \]
\[ f'(x) = 2ax \]

To satisfy \( f(4) = 1 \):
\[ f(4) = 16a = 1 \]
\[ a = \frac{1}{16} \]

There is no value of \( a \) that satisfies requirement (ii)

\[ f'(x) = 2 \cdot 4a = 1 \]
\[ a = \frac{1}{8} \]

Work for problem 3(b)

\[ g(x) = cx^3 - \frac{x^2}{16} \]
\[ g'(x) = 3cx^2 - \frac{x}{8} \]

\[ g(4) = 64c - 1 = 16c \quad c = \frac{1}{32} \]
\[ g'(4) = 48c - \frac{4}{8} = 1 \quad c = \frac{1}{32} \]
Work for problem 3(c)

\[ g(x) = \frac{1}{32}x^3 - \frac{x^2}{16} \]
\[ g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = 0 \]

\[ x\left(\frac{3}{32}x - \frac{1}{8}\right) = 0 \]
\[ x = 0 \quad x = 1.333 \]

Because \( g'(0) = 0 \), \( g(x) \) is not increasing at \( x = 0 \), thus it does not satisfy requirement (ii).

Work for problem 3(d)

\[ h(x) = \frac{x^n}{k} \]
\[ \frac{n}{k} = 1 \quad 4^n = k \]

\[ h'(x) = \frac{n x^{n-1}}{k} \]
\[ \frac{n^4 n^{n-1}}{k} = 1 \quad n^4 n^{n-1} = k \]

\[ 4^n = n^4 n^{n-1} \]

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNLESS YOU ARE TOLD TO DO SO.
Work for problem 3(a)

\[ f(x) = ax^2 \]
\[ y = ax^2 \]
\[ 1 = 16a \]
\[ a = \frac{1}{16} \]
\[ y = \frac{1}{16}x^2 \]

Work for problem 3(b)

\[ x = 4 \]
\[ y = 1 \]
\[ 1 = 64c - 1 \]
\[ 2 = 64c \]
\[ c = \frac{1}{32} \]
Work for problem 3(c)

\[ g(x) = \frac{x^3}{3} - \frac{x^2}{6} \]

\[ = \frac{x^3 - 2x^2}{3} \]

\[
\begin{align*}
\text{x} & = 0 & \text{y} & = 0 \\
\text{x} & = 1 & \text{y} & = -\frac{1}{3} \\
\text{x} & = 2 & \text{y} & = 0 \\
\text{x} & = 3 & \text{y} & = 0
\end{align*}
\]

Work for problem 3(d)

\[ h(x) = \frac{x^n}{k} \]

\[ 1 = \frac{4^n}{k} \]

\[ k = 4^n \]

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.
Question 3

Overview

This problem presented three requirements that had to be satisfied by the graph of a function modeling the height of a skateboard ramp. Students were asked to investigate three families of functions that might be used for such a model. In part (a) they were asked to show that no quadratic of the form $ax^2$ would satisfy the second requirement. In part (b) they were asked to find the coefficient $c$ for which the cubic $cx^3 - \frac{x^2}{16}$ would meet the second requirement, but then show in part (c) that the cubic with this value of $c$ does not meet the third requirement. Finally, in part (d) students were asked to find the values of $n$ and $k$ for which the power function $\frac{x^n}{k}$ would meet all three requirements.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). The student’s work is correct in parts (a) and (b). In part (c) the student earned 1 point for finding the derivative of $g$. The student does not explain why $g$ is not increasing between $x = 0$ and $x = 4$ and so did not earn the second point in this part. In part (d) the student sets up correct equations to find $n$ and $k$, earning 1 point for each equation, but does not find $n$ or $k$ and thus cannot show that the function $h$ meets requirements (i) and (iii).

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (d). In part (a) the student finds the value of $a$ for which $f(4) = 1$, which earned the first point, but fails to show that this value of $a$ does not work to meet requirement (ii). In part (b) the student uses the information about $g$ to find the desired value of $c$. In part (c) the student’s calculations of the values of the function $g$ at integer values of $x$ earned no points (and the value at $x = 3$ is incorrect). However, both points could have been earned in part (c) with those calculations if the student had gone on to observe that the value of $y$ at $x = 1$ is less than the value of $y$ at $x = 0$, and hence the function $g$ is not increasing on the interval $0 \leq x \leq 4$. In part (d) the student earned 1 point for using the information about $h(4)$ to write an equation for $n$ and $k$ but has no other work.