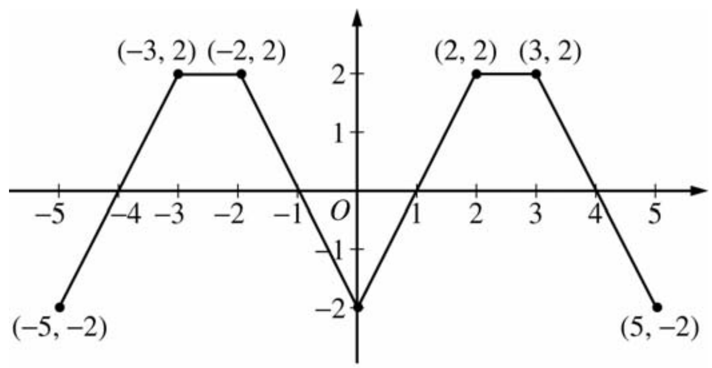
Question 3

The graph of the function f shown above consists of six line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(4)$, $g'(4)$, and $g''(4)$.

(b) Does g have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.



Graph of f

(c) Suppose that f is defined for all real numbers x and is periodic with a period of length 5. The graph above shows two periods of f. Given that $g(5) = 2$, find $g(10)$ and write an equation for the line tangent to the graph of g at $x = 108$.

(a) $g(4) = \int_0^4 f(t)\,dt = 3$

$g'(4) = f(4) = 0$

$g''(4) = f'(4) = -2$

$$3 : \begin{cases} 1 : g(4) \\ 1 : g'(4) \\ 1 : g''(4) \end{cases}$$

(b) g has a relative minimum at $x = 1$ because $g' = f$ changes from negative to positive at $x = 1$.

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$$

(c) $g(0) = 0$ and the function values of g increase by 2 for every increase of 5 in x.

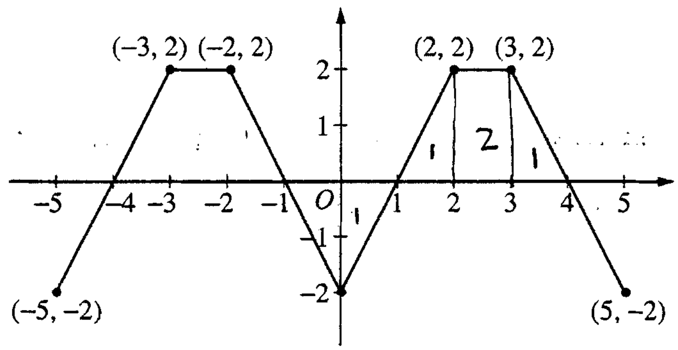
$g(10) = 2g(5) = 4$

$$\begin{aligned} g(108) &= \int_0^{105} f(t)\,dt + \int_{105}^{108} f(t)\,dt \\ &= 21g(5) + g(3) = 44 \end{aligned}$$

$g'(108) = f(108) = f(3) = 2$

An equation for the line tangent to the graph of g at $x = 108$ is $y - 44 = 2(x - 108)$.

$$4 : \begin{cases} 1 : g(10) \\ 3 : \begin{cases} 1 : g(108) \\ 1 : g'(108) \\ 1 : \text{equation of tangent line} \end{cases} \end{cases}$$



Graph of f

Work for problem 3(a)

$g(4) = \int_0^4 f(t)\,dt = 3$

$\boxed{g(4) = 3}$

$\boxed{g'(4) = f(4) = 0}$

$g''(4) = f'(4) = \frac{2+2}{3-5} = \frac{4}{-2} = -2$

$\boxed{g''(4) = f'(4) = -2}$

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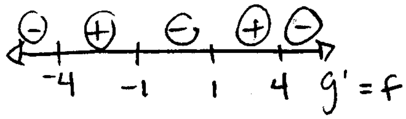
Continue problem 3 on page 9.

Work for problem 3(b)



g has a relative minimum at $x=1$ because $g'(x)=f(x)$ changes from negative to positive at $x=1$

Work for problem 3(c)

$g'(x) = f(x)$
$g(x) = \int f(x)\,dx$

- if $g(5)=2$ and f is periodic w/ a period length of 5, the $\boxed{g(10)=4}$
- $g(108)=?$

$g(108) = \int_0^{108} f(x)\,dx = \boxed{44}$

$g'(108) = f(108) = 2$

$\boxed{(y-44) = 2(x-108)}$

$y = 2x-72$

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END OF PART A OF SECTION II
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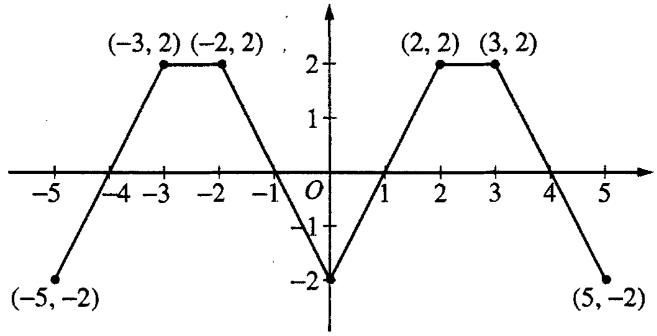

Graph of f

Work for problem 3(a)

$g(4) = \int_0^4 f(t)\,dt = 3$

$g'(4) = \cancel{\frac{d}{dt}\int_0^x} f(t)\,dt = f(x)$

$= f(4) = 0$

$g''(4) = f'(4) = \frac{-2-2}{5-3} = \frac{-4}{2} = -2$

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Continue problem 3 on page 9.

Work for problem 3(b)

$g(1) = \int_0^1 f(x)\,dt = -1$

$g(-5) = \int_0^{-5} f(x)\,dt = -2$

$g(-1) = \int_0^{-1} f(x)\,dt = +1$

$g(4) = \int_0^4 f(x)\,dt = 3$

$g(5) = \int_0^5 f(x)\,dt = 2$

g has neither a max or min @ $x = 1$ because max occurs @ $x = 4$ and min @ $x = -5$

Work for problem 3(c)

$\frac{10}{5} = 2 \qquad g(5) = \int_0^5 f(t)\,dt = 2$

$g(10) = 2\int_0^5 f(t)\,dt$

$g(10) = 2(2) = 4$

$g(108) = \frac{108}{5}\int_0^5 f(t)\,dt = \frac{216}{5}$

$\frac{108}{5} = 21\,\frac{3}{5} \qquad f(108) = f(3)$

$g'(108) = f(108) = f(3) = 2$

$y - \frac{216}{5} = 2(x - 108)$

END OF PART A OF SECTION II
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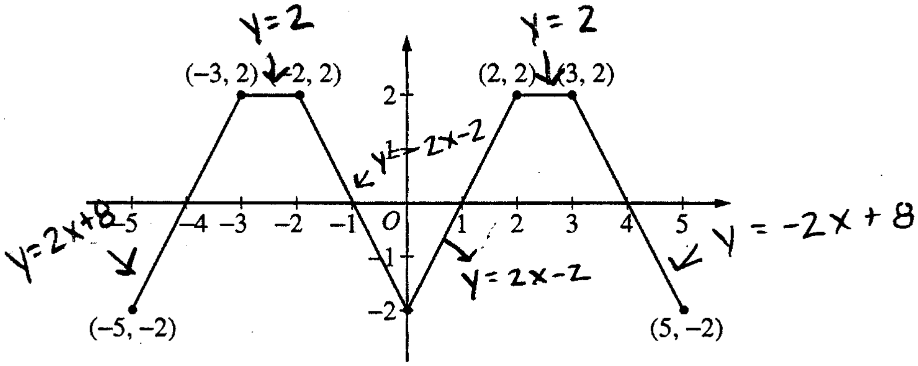

Graph of f

Work for problem 3(a)

$\frac{2-(-2)}{-3+5} = 2 \quad y = 2x+8 \qquad \frac{2-(-2)}{-2} = -2 \quad y = -2x-2$

$\frac{2+2}{2} = 2 \quad y = 2x-2 \qquad \frac{2+2}{3-5} = -2 \quad y = -2x+8$

$g(4) = \int_0^4 (-2x+8)\,dx$

$= -x^2 + 8x \Big]_0^4$

$= 16$

$\boxed{\therefore g(4) = 16}$

$\boxed{g'(4) = 0}$

$\boxed{g''(4) = -2}$

Continue problem 3 on page 9.

 3C2

Work for problem 3(b)

− o +|+ o − o + o
−4 −1 1 4

There is a relative minimum at $x = 1$ because the sign analysis shows that slope is negative from $(-1, 1)$ and then there is 0 slope at $x = 1$ and slope is positive from $(1, 4)$.

Work for problem 3(c)

slope $= -2$

$2 = -2(5) + b$

$b = 12$

$\therefore y = -2x + 12$

The equation of the line tangent to the curve is $y = -2x + 12$

$y|_{x=10} = -8$

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END OF PART A OF SECTION II

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Question 3

Overview

This problem required the Fundamental Theorem of Calculus. Students were given the piecewise-linear graph of the function f and were asked about the function g defined as the definite integral of f from 0 to x. It was expected that students would use the graph of f, as well as the area bounded by the graph of f and the x-axis, to answer questions about g, g', and g''. Part (a) asked for the values of $g(4)$, $g'(4)$, and $g''(4)$. Part (b) asked about the behavior of g at $x = 1$. In part (c) the function f is extended in a periodic fashion. Students had to compute $g(10)$, $g(108)$, and $g'(108)$ using the periodic behavior of f.

Sample: 3A
Score: 9

The student earned all 9 points. In part (c) the student writes an equation $y = 2x - 72$ but declares the correct answer for the equation of the tangent line by enclosing it in a box.

Sample: 3B
Score: 6

The student earned 6 points: 3 points in part (a) and 3 points in part (c). In part (a) the student gives correct values for $g(4)$, $g'(4)$, and $g''(4)$, which earned 3 points. In part (b) the student claims that there is neither a maximum nor a minimum at $x = 1$; consequently, the student was not eligible for the justification point. In part (c) the student earned 1 point for giving the correct value for $g(10)$. The student declares an incorrect value for $g(108)$ but a correct value for $g'(108)$. These values are correctly used to write the equation of the tangent line. The student earned 1 point for $g'(108)$ and 1 point for the tangent line since the incorrect value for $g(108)$ is used correctly.

Sample: 3C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (b). In part (a) the student correctly identifies $g'(4)$ and $g''(4)$, which earned 2 points. In part (b) the student declares a relative minimum at $x = 1$ but did not earn the justification point. The argument refers to slope without designating whether it refers to the slope of f or the slope of g. No points were earned in part (c).