The graph of the function $f$ shown above consists of six line segments. Let $g$ be the function given by

$$g(x) = \int_0^x f(t) \, dt.$$ 

(a) Find $g(4)$, $g'(4)$, and $g''(4)$.

(b) Does $g$ have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.

(c) Suppose that $f$ is defined for all real numbers $x$ and is periodic with a period of length 5. The graph above shows two periods of $f$. Given that $g(5) = 2$, find $g(10)$ and write an equation for the line tangent to the graph of $g$ at $x = 108$.

<table>
<thead>
<tr>
<th>(a) $g(4) = \int_0^4 f(t) , dt = 3$</th>
<th>$g'(4) = f(4) = 0$</th>
<th>$g''(4) = f'(4) = -2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b) $g$ has a relative minimum at $x = 1$</td>
<td>because $g' = f$ changes from negative to positive at $x = 1$.</td>
<td>2 : 1 : answer</td>
</tr>
<tr>
<td>(c) $g(0) = 0$ and the function values of $g$ increase by 2 for every increase of 5 in $x$.</td>
<td>$g(10) = 2g(5) = 4$</td>
<td>$g(108) = \int_0^{108} f(t) , dt + \int_{105}^{108} f(t) , dt$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 21g(5) + g(3) = 44$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$g'(108) = f(108) = f(3) = 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>An equation for the line tangent to the graph of $g$ at $x = 108$ is $y - 44 = 2(x - 108)$.</td>
</tr>
</tbody>
</table>
Work for problem 3(a)

\[ g'(4) = f(4) = 0 \]

\[ g''(4) = f'(4) = \frac{2 + 2}{3 - 5} = \frac{4}{-2} = -2 \]

\[ g''(4) = f'(4) = -2 \]
Work for problem 3(b)

\[ g \] has a relative minimum at \( x = 1 \) because \( g'(x) = f(x) \) changes from negative to positive at \( x = 1 \).

\[
\begin{array}{cccc}
-4 & -1 & 1 & 4 \\
\hline
\circ & + & \circ & \circ
\end{array}
\]

\[ g' = f \]

Work for problem 3(c)

\[ g'(x) = f(x) \]
\[ g(x) = \int f(x) \, dx \]

- If \( g(5) = 2 \) and \( f \) is periodic with a period length of 5, then \( g(10) = 4 \).

- \( g(108) = ? \)

\[
g(108) = \int_{0}^{108} f(x) \, dx = 44
\]

- \[ g'(108) = f(108) = 2 \]

\[
\left( y - 44 \right) = 2 \left( x - 108 \right)
\]

\[ y = 2x - 72 \]

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.
Work for problem 3(a)

\[ g(4) = \int_0^4 f(t) \, dt = 3 \]

\[ g'(4) = \frac{d}{dx} \int_0^x f(t) \, dt = f(x) \]

\[ = f(4) = 0 \]

\[ g''(4) = f'(4) = \frac{-2 - 2}{5 - 3} = \frac{-4}{2} = -2 \]

Continue problem 3 on page 9.
Work for problem 3(b)

\[ g(1) = \int_{0}^{1} f(t) \, dt = -1 \]
\[ g(-5) = \int_{0}^{-5} f(x) \, dx = -2 \]
\[ g(-1) = \int_{0}^{-1} f(x) \, dx = 1 \]
\[ g(4) = \int_{0}^{4} f(x) \, dx = 3 \]
\[ g(5) = \int_{0}^{5} f(x) \, dx = 2 \]

\[ g \text{ has neither a max or min} \]
\[ \Theta \quad x = 1 \text{ because max occurs} \]
\[ \Theta \quad x = 4 \text{ and min } \Theta \]
\[ x = -5 \]

Work for problem 3(c)

\[ \frac{10}{5} = 2 \]
\[ g(5) = \int_{0}^{5} f(t) \, dt = 2 \]

\[ g(10) = 2 \int_{0}^{5} f(t) \, dt \]
\[ g(10) = 2 \cdot 2 = 4 \]

\[ g(108) = \frac{108}{5} \int_{0}^{5} f(t) \, dt = \frac{216}{5} \]

\[ \frac{108}{5} = 21 \quad \Theta \]
\[ f(108) = f(3) \]

\[ g'(108) = f(108) = f(3) = 2 \]
\[ y - \frac{216}{5} = 2(x - 108) \]

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.
Work for problem 3(a)

\[
\frac{2 - (-2)}{-3 + 5} = 2 \quad \gamma = 2x + 8
\]

\[
\frac{2 - (-2)}{-2} = -2 \quad \gamma = -2x - 2
\]

\[
\frac{2 + 2}{2} = 2 \quad \gamma = 2x - 2
\]

\[
\frac{2 + 2}{3 - 5} = -2 \quad \gamma = -2x + 8
\]

\[g(4) = \int_{6}^{4} (-2x + 8) \, dx\]

\[= -x^2 + 8x\bigg|_{6}^{4}\]

\[= 16 - 36 = -20\]

\[\therefore g(4) = 16\]

\[g'(4) = 0\]

\[g''(4) = -2\]
Work for problem 3(b)

There is a relative minimum at $x = 1$ because the sign analysis shows that slope is negative from $(-1, 1)$ and then there is no slope at $x = 1$ and slope is positive from $(1, 4)$.

Work for problem 3(c)

$$\text{slope} = -2$$

$$2 = -2(5) + b$$

$$b = 12$$

$$y = -2x + 12$$

The equation of the line tangent to the curve is $y = -2x + 12$

$$y\big|_{x=10} = -8$$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.
Question 3

Overview

This problem required the Fundamental Theorem of Calculus. Students were given the piecewise-linear graph of the function $f$ and were asked about the function $g$ defined as the definite integral of $f$ from 0 to $x$. It was expected that students would use the graph of $f$, as well as the area bounded by the graph of $f$ and the $x$-axis, to answer questions about $g$, $g'$, and $g''$. Part (a) asked for the values of $g(4)$, $g'(4)$, and $g''(4)$. Part (b) asked about the behavior of $g$ at $x = 1$. In part (c) the function $f$ is extended in a periodic fashion. Students had to compute $g(10)$, $g(108)$, and $g'(108)$ using the periodic behavior of $f$.

Sample: 3A
Score: 9

The student earned all 9 points. In part (c) the student writes an equation $y = 2x - 72$ but declares the correct answer for the equation of the tangent line by enclosing it in a box.

Sample: 3B
Score: 6

The student earned 6 points: 3 points in part (a) and 3 points in part (c). In part (a) the student gives correct values for $g(4)$, $g'(4)$, and $g''(4)$, which earned 3 points. In part (b) the student claims that there is neither a maximum nor a minimum at $x = 1$; consequently, the student was not eligible for the justification point. In part (c) the student earned 1 point for giving the correct value for $g(10)$. The student declares an incorrect value for $g(108)$ but a correct value for $g'(108)$. These values are correctly used to write the equation of the tangent line. The student earned 1 point for $g'(108)$ and 1 point for the tangent line since the incorrect value for $g(108)$ is used correctly.

Sample: 3C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (b). In part (a) the student correctly identifies $g'(4)$ and $g''(4)$, which earned 2 points. In part (b) the student declares a relative minimum at $x = 1$ but did not earn the justification point. The argument refers to slope without designating whether it refers to the slope of $f$ or the slope of $g$. No points were earned in part (c).