Question 6

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	–20	–30	–20	–14	–10	0	10
$a(t)$ $\left(\text{ft}/\text{sec}^2\right)$	1	5	2	1	2	4	2

A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity v, measured in feet per second, and acceleration a, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

(a) Using appropriate units, explain the meaning of $\int_{30}^{60} |v(t)|\,dt$ in terms of the car's motion. Approximate $\int_{30}^{60} |v(t)|\,dt$ using a trapezoidal approximation with the three subintervals determined by the table.

(b) Using appropriate units, explain the meaning of $\int_{0}^{30} a(t)\,dt$ in terms of the car's motion. Find the exact value of $\int_{0}^{30} a(t)\,dt$.

(c) For $0 < t < 60$, must there be a time t when $v(t) = -5$? Justify your answer.

(d) For $0 < t < 60$, must there be a time t when $a(t) = 0$? Justify your answer.

(a) $\int_{30}^{60} \lvert v(t)\rvert\,dt$ is the distance in feet that the car travels from $t = 30$ sec to $t = 60$ sec. Trapezoidal approximation for $\int_{30}^{60} \lvert v(t)\rvert\,dt$: $A = \frac{1}{2}(14+10)5 + \frac{1}{2}(10)(15) + \frac{1}{2}(10)(10) = 185$ ft	$2: \begin{cases} 1: \text{explanation} \\ 1: \text{value} \end{cases}$
(b) $\int_{0}^{30} a(t)\,dt$ is the car's change in velocity in ft/sec from $t = 0$ sec to $t = 30$ sec. $\int_{0}^{30} a(t)\,dt = \int_{0}^{30} v'(t)\,dt = v(30) - v(0)$ $= -14 - (-20) = 6$ ft/sec	$2: \begin{cases} 1: \text{explanation} \\ 1: \text{value} \end{cases}$
(c) Yes. Since $v(35) = -10 < -5 < 0 = v(50)$, the IVT guarantees a t in $(35, 50)$ so that $v(t) = -5$.	$2: \begin{cases} 1: v(35) < -5 < v(50) \\ 1: \text{Yes; refers to IVT or hypotheses} \end{cases}$
(d) Yes. Since $v(0) = v(25)$, the MVT guarantees a t in $(0, 25)$ so that $a(t) = v'(t) = 0$.	$2: \begin{cases} 1: v(0) = v(25) \\ 1: \text{Yes; refers to MVT or hypotheses} \end{cases}$
Units of ft in (a) and ft/sec in (b)	1 : units in (a) and (b)

NO CALCULATOR ALLOWED

6A1

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	–20	–30	–20	–14	–10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

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Work for problem 6(a)

$\int_{30}^{60} |v(t)|\,dt$ is the total distance, in feet, travelled by the car from 30 to 60 seconds.

$$\int_{30}^{60} |v(t)|\,dt = \left|\tfrac{1}{2}(5)(-14-10)\right| + \left|\tfrac{1}{2}(15)(-10+0)\right| + \tfrac{1}{2}(10+0)(10)$$

$$= \frac{(24)(5)}{2} + \frac{(10)(15)}{2} + \frac{(10)(10)}{2}$$

$$= 60 + 75 + 50 = 185 \text{ feet.}$$

Work for problem 6(b)

$\int_{0}^{30} a(t)\,dt$ is the change in velocity from 0 seconds to 30 seconds (in feet/sec).

$$\int_{0}^{30} a(t)\,dt = v(30) - v(0) = -14-(-20) = 6 \text{ ft/sec}$$

$\therefore$ there was an overall increase in the velocity of the car during the first 30 seconds equal to 6 ft/sec.

Continue problem 6 on page 15

6A2

Work for problem 6(c)

Yes, there must

Since $v(t)$ is continuous over $[35, 50]$ & differentiable,

& $v(t)$ is increasing & $a(t)$ is also increasing

during this time interval,

& since $v(t) = -5 \in (\overset{-10}{v(35)}, \overset{0}{v(50)}] \Rightarrow$

$\exists$ a time $t \in (35, 50) \mid v(t) = -5$ ft/sec

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Work for problem 6(d)

Yes, there must.

for $0 < t < 60$, $\exists$ a $t \in (0, 60) \mid a(t) = 0$

by the mean value theorem

$v(0) = v(25) = -20$ ft/sec

$a(t) = v'(t) \Rightarrow$ since $v(t)$ is continuous & differentiable for $(0, 25) \Rightarrow$

$$\exists\ t \in (0,25) \mid v'(t) = \frac{v(25) - v(0)}{25 - 0} = 0$$

$\therefore a(t) = 0$

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

6B₁

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	–20	–30	–20	–14	–10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

Work for problem 6(a)

$\int_{30}^{60} |v(t)|\,dt$ represents the total distance in feet travelled by the car between $t = 30$ and 60 seconds.

$$A_{tot.} = \left|\left(\frac{v(30)+v(35)}{2}\right)(35-30)\right| + \left|\left(\frac{v(50)+v(35)}{2}\right)(50-35)\right| + \left|\left(\frac{v(60)+v(50)}{2}\right)(60$$

$$= \left|\left(\frac{-10-14}{2}\right)(5)\right| + \left|\left(\frac{-10}{2}(15)\right)\right| + \left|\frac{10}{2}(10)\right|$$

$$= 60 + 65 + 50 = \underline{175 \text{ feet.}}$$

Work for problem 6(b)

$\int_0^{30} a(t)\,dt$ represents the velocity of the car at $t = 30$ seconds in feet per second.

$$\underline{v(30) = -14 \text{ ft/s}}$$

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Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

6B2

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Work for problem 6(c)

Yes; since $V(t)$ is a continuous function, it must pass through every value between -10 and 0 between $t = 35$ and $t = 50$

Work for problem 6(d)

Yes. Using Rolle's theorum, one finds that if the slope of a line between two points on a continuous function equals zero, than there must be some point, c, where the slope of that function is equal to zero.

$$\frac{-20-(-20)}{25-0} = 0$$

The line between the points at $t=0$ and $t=25$ is horizontal. It has zero slope so $a(t)$ (which equals $v'(t)$) must equal zero at some point between $t=0$ and $t=25$

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NO CALCULATOR ALLOWED

6C1

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	−20	−30	−20	−14	−10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

Work for problem 6(a)

a) $\int_{30}^{60} |v(t)|\,dt$ is the total displacement of the car from 30 sec to 60 sec.

$$\int_{30}^{60} |v(t)|\,dt = \frac{(-14+-10)(5)}{2} + \frac{(-10+0)(15)}{2} + \frac{(10+0)(10)}{2} = -60 + -75 + 50$$
$$= -85$$

-85 ft

Work for problem 6(b)

b) $\int_{0}^{30} a(t)\,dt$ is the average velocity of the car from 0 sec to 30 sec.

$$\int_{0}^{30} a(t)\,dt = -14 - -20 = 6.$$

6 ft/sec

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Continue problem 6 on page 15.

6C2

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Work for problem 6(c)

c) yes, because of the intermediate value theorum. from $t=35$ to $t=50$ the velocity changes from -10 to 0, that means the velocity had to be at -5 some time between 35 and 50 because it is a continuous function.

Work for problem 6(d)

d) no, because the function is continuous and because of the intermediate value theorum, the acceleration was never at zero, because all the acceleration values are greater than zero.

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Question 6

Overview

This problem presented students with a table of the values of a car's velocity and acceleration at selected times. In part (a) students had to recognize the given definite integral as the total distance traveled by the car, in feet, from time $t = 30$ seconds to time $t = 60$ seconds and then approximate this distance using a trapezoidal approximation with three intervals of unequal lengths. In part (b) students had to recognize the given definite integral as the total change in velocity, in feet per second, from time $t = 0$ seconds to time $t = 30$ seconds and then calculate the exact value of this integral using the Fundamental Theorem of Calculus. Units of measure were important in both parts (a) and (b). In part (c) students were expected to use the Intermediate Value Theorem with $v(t)$ to justify that $v(t) = -5$ somewhere on the interval. Part (d) asked a similar question about the acceleration, but here students were expected to use the Mean Value Theorem applied to $v(t)$ to show the existence of a time t when $a(t) = v'(t) = 0$.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (c), 2 points in part (d), and the units point. In part (a) the student earned the explanation point but makes an arithmetic mistake in the last line and so did not earn the value point. In part (b) the student understands that velocity is the antiderivative of acceleration but does not recognize the definite integral as the change in the velocity and did not earn the explanation point. The student does not find the correct value for the definite integral. In part (c) the student makes the correct conclusion and gives a correct reason, which earned both points. It is not necessary to name the Intermediate Value Theorem since the hypothesis (continuity) is mentioned. In part (d) the student applies Rolle's Theorem (the Mean Value Theorem is also acceptable). Although the student only mentions continuity in the general description of Rolle's Theorem, the second point was still earned. The correct units are used in parts (a) and (b), and the student therefore earned the units point.

Sample: 6C
Score: 4

The student earned 4 points: 1 point in part (b), 2 points in part (c), and the units point. In part (a) the "total displacement" is not the same as the total distance traveled, and so the student did not earn the first point. Because the student fails to use the absolute value in the first two terms of the trapezoidal approximation, the answer is incorrect. In part (b) the integral is not the "average velocity" so the student did not earn the first point. The computation for the second point is correct. In part (c) the student earned both points by correctly citing the Intermediate Value Theorem and drawing the correct conclusion. In part (d) the student answers "no." There is no way to justify an incorrect answer so the student earned no points in this part. The student gives the correct units in (a) and (b) and earned the units point.