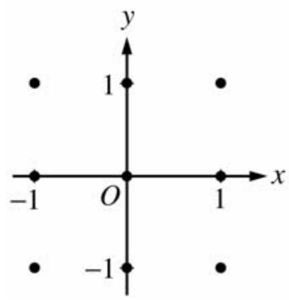
Question 5

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

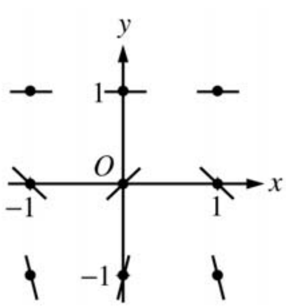
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

(a)



$2: \begin{cases} 1: \text{zero slopes} \\ 1: \text{all other slopes} \end{cases}$

(b) The line $y = 1$ satisfies the differential equation, so $c = 1$.

$1: c = 1$

(c) $\frac{1}{(y-1)^2}\, dy = \cos(\pi x)\, dx$

$-(y-1)^{-1} = \frac{1}{\pi}\sin(\pi x) + C$

$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + C$

$1 = \frac{1}{\pi}\sin(\pi) + C = C$

$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + 1$

$\frac{\pi}{1-y} = \sin(\pi x) + \pi$

$y = 1 - \frac{\pi}{\sin(\pi x) + \pi}$ for $-\infty < x < \infty$

$6: \begin{cases} 1: \text{separates variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{answer} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

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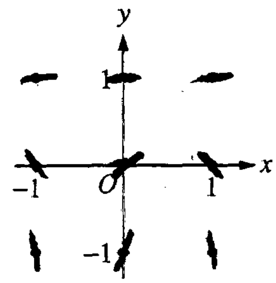
NO CALCULATOR ALLOWED

Work for problem 5(a)

$\frac{dy}{dx} = (y-1)^2 \cdot \cos \pi x.$



Work for problem 5(b)

$y = C$

The horizontal line is $y=1$, because where $y=1$ the derivative of the the function is zero (it does not depend what x we take). And as equation of the linear function is $y=bx+c$ and the slope of it is zero, we may say that $\underline{y=1}$.

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NO CALCULATOR ALLOWED

Work for problem 5(c)

$$\begin{cases} \dfrac{dy}{dx} = (y-1)^2 \cdot \cos(\pi x) \\ f(1) = 0. \end{cases}$$

$$\int \frac{dy}{(y-1)^2} = \int \cos(\pi x)\, dx$$

$$-\frac{1}{y-1} = \frac{\sin(\pi x)}{\pi} + C.$$

$$-\frac{1}{0-1} = \frac{\sin \pi}{\pi} + C.$$

$$1 = 0 + C.$$

$$\underline{C = 1}.$$

$$-\frac{1}{y-1} = \frac{\sin(\pi x)}{\pi} + 1.$$

$$\frac{1}{y-1} = -\frac{\sin(\pi x)}{\pi} - 1.$$

$$y-1 = -\frac{1}{\frac{\sin(\pi x)}{\pi} + 1}$$

$$\underline{y = -\frac{\pi}{\sin(\pi x) + \pi} + 1}.$$

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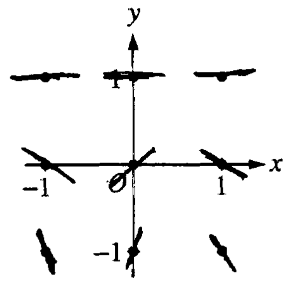
Work for problem 5(a)



$(0-1)^2 \cos(\pi(0)) = 1$

$(-1-1)^2 \cos(\pi(0)) = 4$

$(0-1)^2 \cos\pi = -1$

$(1-1)^2 \cos\pi = 0$

$(0-1)^2 \cos -\pi = -1$

$(-1-1)^2 \cos -\pi = -4$

$(-1-1)^2 \cos\pi = -4$

Work for problem 5(b)

$\frac{dy}{dx} = (y-1)^2 \cos\pi x$ $\quad \int \frac{1}{(y-1)^2}\frac{dy}{dx} = \int \cos\pi x \, dx$

$-\frac{1}{y-1} = \frac{\sin\pi x}{\pi}$ $\quad y = -\frac{\pi}{\sin\pi x} + 1$

$y = 1$ $\boxed{C = 1}$

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5B

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Work for problem 5(c)

$u = y-1 \quad dx = \frac{du}{1}$

$\frac{dy}{dx} = (y-1)^2 \cos \pi x \qquad \int \frac{1}{(y-1)^2} \frac{dy}{dx} = \int \cos \pi x \, dx$

$-\frac{1}{y-1} = \frac{\sin \pi x}{\pi} \qquad y = -\frac{\pi}{\sin \pi x} + 1 + C$

$0 = \frac{-\pi}{\sin \pi} + 1 + C \qquad C = 0$

$\boxed{y = \frac{-\pi}{\sin \pi x}}$

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5C

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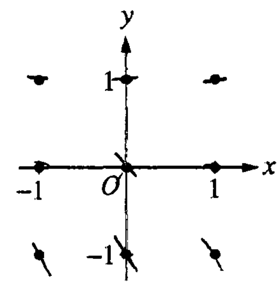
Work for problem 5(a)



Work for problem 5(b)

$y' = 0$

$(y-1)^2 \cos(\pi x) = 0$

$=$

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NO CALCULATOR ALLOWED

Work for problem 5(c)

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$\frac{1}{(y-1)^2} dy = \cos(\pi x) \cdot dx$

$\frac{-2}{y-1} = \sin(\pi x)(\pi) + C$

$\frac{-2}{(0)-1} = \sin(\pi(1))(\pi) + C$

$-2 - \pi \sin \pi = C$

$-2 = C$

$y - 1 = \frac{-2}{\pi \sin \pi - 2}$

$y = \frac{-2}{\pi \sin \pi - 2} + 1$ //

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Question 5

Overview

This problem presented students with a separable differential equation. In part (a) students were asked to sketch its slope field at nine points. In part (b) students needed to recognize that if there is a horizontal line with equation $y = c$ that satisfies the differential equation, then $y = c$ must make $\frac{dy}{dx} = 0$ for all values of x. Part (c) required solving the separable differential equation to find the particular solution with $f(1) = 0$.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). The work in parts (a) and (b) is correct. In part (c) the student correctly separates variables and finds the two antiderivatives, which earned 3 points. The student does not have a constant of integration so did not earn any of the last 3 points. The student does eventually add a constant but only after doing some algebraic simplification and thus at an inappropriate step in trying to find the particular solution.

Sample: 5C
Score: 4

The student earned 4 points: 1 point in part (a) and 3 points in part (c). In part (a) the zero slopes are correct, which earned 1 point. The slopes on the x-axis and at $(0, -1)$ are incorrect or missing. In part (b) the student does not complete the work and earned no points. In part (c) the student correctly separates variables and earned the first point. Both antiderivatives are incorrect. The student earned the third and fourth points with a correct introduction of a constant of integration and use of the initial condition. The sixth point was not earned because the student makes an error in solving for C.

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