CollegeBoard
Advanced Placement Program

AP® Calculus BC
2004 Scoring Guidelines
Form B

Question 1

A particle moving along a curve in the plane has position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sqrt{t^4 + 9} \text{ and } \frac{dy}{dt} = 2e^t + 5e^{-t}$$

for all real values of t. At time $t = 0$, the particle is at the point $(4, 1)$.

(a) Find the speed of the particle and its acceleration vector at time $t = 0$.

(b) Find an equation of the line tangent to the path of the particle at time $t = 0$.

(c) Find the total distance traveled by the particle over the time interval $0 \le t \le 3$.

(d) Find the x-coordinate of the position of the particle at time $t = 3$.

(a) At time $t = 0$:

Speed $= \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{3^2 + 7^2} = \sqrt{58}$

Acceleration vector $= \langle x''(0), y''(0) \rangle = \langle 0, -3 \rangle$

$2 : \begin{cases} 1 : \text{speed} \\ 1 : \text{acceleration vector} \end{cases}$

(b) $\dfrac{dy}{dx} = \dfrac{y'(0)}{x'(0)} = \dfrac{7}{3}$

Tangent line is $y = \frac{7}{3}(x - 4) + 1$

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \end{cases}$

(c) Distance $= \displaystyle\int_0^3 \sqrt{\left(\sqrt{t^4 + 9}\right)^2 + \left(2e^t + 5e^{-t}\right)^2}\, dt$

$= 45.226$ or 45.227

$3 : \begin{cases} 2 : \text{distance integral} \\ \quad \langle -1 \rangle \text{ each integrand error} \\ \quad \langle -1 \rangle \text{ error in limits} \\ 1 : \text{answer} \end{cases}$

(d) $x(3) = 4 + \displaystyle\int_0^3 \sqrt{t^4 + 9}\, dt$

$= 17.930$ or 17.931

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Question 2

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

(a) $f(2) = T(2) = 7$ $\dfrac{f''(2)}{2!} = -9$ so $f''(2) = -18$	2 : { 1 : $f(2) = 7$ 1 : $f''(2) = -18$
(b) Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$. Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.	2 : { 1 : states $f'(2) = 0$ 1 : declares $f(2)$ as a relative maximum because $f''(2) < 0$
(c) $f(0) \approx T(0) = -5$ It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.	3 : { 1 : $f(0) \approx T(0) = -5$ 1 : declares that it is not possible to determine 1 : reason
(d) Lagrange error bound $= \dfrac{6}{4!}\lvert 0 - 2\rvert^4 = 4$ $f(0) \le T(0) + 4 = -1$ Therefore, $f(0)$ is negative.	2 : { 1 : value of Lagrange error bound 1 : explanation

Question 3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

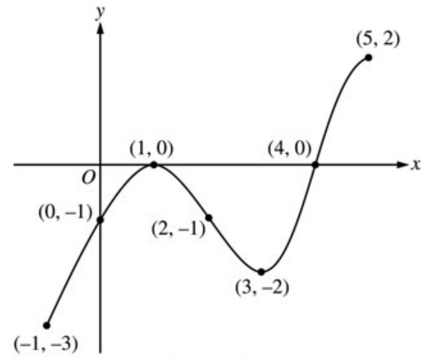
(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicates units of measure.

(d) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

(a) Midpoint Riemann sum is $10\cdot[v(5) + v(15) + v(25) + v(35)]$ $= 10\cdot[9.2 + 7.0 + 2.4 + 4.3] = 229$. The integral gives the total distance in miles that the plane flies during the 40 minutes.	3 : { 1 : $v(5) + v(15) + v(25) + v(35)$; 1 : answer; 1 : meaning with units }
(b) By the Mean Value Theorem, $v'(t) = 0$ somewhere in the interval $(0, 15)$ and somewhere in the interval $(25, 30)$. Therefore the acceleration will equal 0 for at least two values of t.	2 : { 1 : two instances; 1 : justification }
(c) $f'(23) = -0.407$ or -0.408 miles per minute^2	1 : answer with units
(d) Average velocity $= \frac{1}{40}\int_0^{40} f(t)\,dt$ $= 5.916$ miles per minute	3 : { 1 : limits; 1 : integrand; 1 : answer }

Question 4



Graph of f'

The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \le x \le 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $f(2) = 6$.

(a) Find the x-coordinate of each of the points of inflection of the graph of f. Give a reason for your answer.

(b) At what value of x does f attain its absolute minimum value on the closed interval $-1 \le x \le 5$? At what value of x does f attain its absolute maximum value on the closed interval $-1 \le x \le 5$? Show the analysis that leads to your answers.

(c) Let g be the function defined by $g(x) = xf(x)$. Find an equation for the line tangent to the graph of g at $x = 2$.

(a) $x = 1$ and $x = 3$ because the graph of f' changes from increasing to decreasing at $x = 1$, and changes from decreasing to increasing at $x = 3$.

$2 : \begin{cases} 1 : x = 1,\ x = 3 \\ 1 : \text{reason} \end{cases}$

(b) The function f decreases from $x = -1$ to $x = 4$, then increases from $x = 4$ to $x = 5$.
Therefore, the absolute minimum value for f is at $x = 4$.
The absolute maximum value must occur at $x = -1$ or at $x = 5$.

$$f(5) - f(-1) = \int_{-1}^{5} f'(t)\, dt < 0$$

Since $f(5) < f(-1)$, the absolute maximum value occurs at $x = -1$.

$4 : \begin{cases} 1 : \text{indicates } f \text{ decreases then increases} \\ 1 : \text{eliminates } x = 5 \text{ for maximum} \\ 1 : \text{absolute minimum at } x = 4 \\ 1 : \text{absolute maximum at } x = -1 \end{cases}$

(c) $g'(x) = f(x) + xf'(x)$
$g'(2) = f(2) + 2f'(2) = 6 + 2(-1) = 4$
$g(2) = 2f(2) = 12$

Tangent line is $y = 4(x - 2) + 12$

$3 : \begin{cases} 2 : g'(x) \\ 1 : \text{tangent line} \end{cases}$

Question 5

Let g be the function given by $g(x) = \frac{1}{\sqrt{x}}$.

(a) Find the average value of g on the closed interval $[1, 4]$.

(b) Let S be the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis. Find the volume of S.

(c) For the solid S, given in part (b), find the average value of the areas of the cross sections perpendicular to the x-axis.

(d) The average value of a function f on the unbounded interval $[a, \infty)$ is defined to be $\lim_{b\to\infty}\left[\frac{\int_a^b f(x)\,dx}{b-a}\right]$. Show that the improper integral $\int_4^\infty g(x)\,dx$ is divergent, but the average value of g on the interval $[4, \infty)$ is finite.

(a) $\frac{1}{3}\int_1^4 \frac{1}{\sqrt{x}}\,dx = \frac{1}{3}\cdot 2\sqrt{x}\Big|_1^4 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

2 : { 1 : integral; 1 : antidifferentiation and evaluation }

(b) Volume $= \pi\int_1^4 \frac{1}{x}\,dx = \pi \ln x\Big|_1^4 = \pi \ln 4$

2 : { 1 : integral; 1 : antidifferentiation and evaluation }

(c) The cross section at x has area $\pi\left(\frac{1}{\sqrt{x}}\right)^2 = \frac{\pi}{x}$

Average value $= \frac{1}{3}\int_1^4 \frac{\pi}{x}\,dx = \frac{1}{3}\pi \ln 4$

1 : answer

(d) $\int_4^\infty g(x)\,dx = \lim_{b\to\infty}\int_4^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left(2\sqrt{b} - 4\right) = \infty$

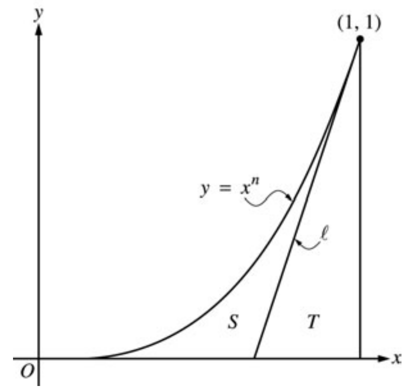
This limit is not finite, so the integral is divergent.

$$\frac{\int_4^b g(x)\,dx}{b-4} = \frac{1}{b-4}\int_4^b \frac{1}{\sqrt{x}}\,dx = \frac{2\sqrt{b}-4}{b-4}$$

$$\lim_{b\to\infty}\frac{2\sqrt{b}-4}{b-4} = 0$$

4 : { 1 : $\int_4^b g(x)\,dx = 2\sqrt{b} - 4$; 1 : indicates integral diverges; 1 : $\frac{1}{b-4}\int_4^b g(x)\,dx = \frac{2\sqrt{b}-4}{b-4}$; 1 : finite limit as $b \to \infty$ }

Question 6



Let ℓ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.

(a) Find $\int_0^1 x^n\,dx$ in terms of n.

(b) Let T be the triangular region bounded by ℓ, the x-axis, and the line $x = 1$. Show that the area of T is $\frac{1}{2n}$.

(c) Let S be the region bounded by the graph of $y = x^n$, the line ℓ, and the x-axis. Express the area of S in terms of n and determine the value of n that maximizes the area of S.

(a) $\int_0^1 x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$

2 : $\begin{cases} 1 : \text{antiderivative of } x^n \\ 1 : \text{answer} \end{cases}$

(b) Let b be the length of the base of triangle T.

$\frac{1}{b}$ is the slope of line ℓ, which is n

$\text{Area}(T) = \frac{1}{2}b(1) = \frac{1}{2n}$

3 : $\begin{cases} 1 : \text{slope of line } \ell \text{ is } n \\ 1 : \text{base of } T \text{ is } \frac{1}{n} \\ 1 : \text{shows area is } \frac{1}{2n} \end{cases}$

(c) $\text{Area}(S) = \int_0^1 x^n\,dx - \text{Area}(T)$

$= \frac{1}{n+1} - \frac{1}{2n}$

$\frac{d}{dn}\text{Area}(S) = -\frac{1}{(n+1)^2} + \frac{1}{2n^2} = 0$

$2n^2 = (n+1)^2$

$\sqrt{2}\,n = (n+1)$

$n = \frac{1}{\sqrt{2}-1} = 1 + \sqrt{2}$

4 : $\begin{cases} 1 : \text{area of } S \text{ in terms of } n \\ 1 : \text{derivative} \\ 1 : \text{sets derivative equal to 0} \\ 1 : \text{solves for } n \end{cases}$