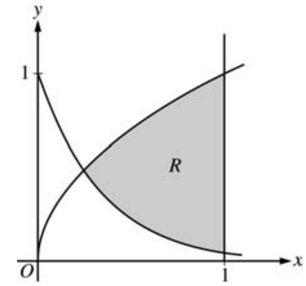
AP® Calculus BC
2003 Scoring Guidelines

Question 1

Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.



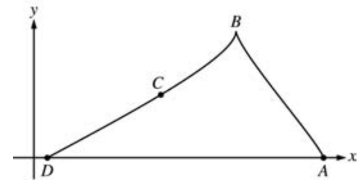
(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in region R. Find the volume of this solid.

Point of intersection $e^{-3x} = \sqrt{x}$ at $(T, S) = (0.238734, 0.488604)$	1: Correct limits in an integral in (a), (b), or (c)
(a) Area $= \int_T^1 \left(\sqrt{x} - e^{-3x}\right) dx$ $= 0.442$ or 0.443	$2: \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) Volume $= \pi \int_T^1 \left(\left(1 - e^{-3x}\right)^2 - \left(1 - \sqrt{x}\right)^2\right) dx$ $= 0.453\pi$ or 1.423 or 1.424	$3: \begin{cases} 2 : \text{integrand} \\ \quad < -1 > \ \text{reversal} \\ \quad < -1 > \ \text{error with constant} \\ \quad < -1 > \ \text{omits 1 in one radius} \\ \quad < -2 > \ \text{other errors} \\ 1 : \text{answer} \end{cases}$
(c) Length $= \sqrt{x} - e^{-3x}$ Height $= 5\left(\sqrt{x} - e^{-3x}\right)$ Volume $= \int_T^1 5\left(\sqrt{x} - e^{-3x}\right)^2 dx = 1.554$	$3: \begin{cases} 2 : \text{integrand} \\ \quad < -1 > \ \text{incorrect but has } \sqrt{x} - e^{-3x} \text{ as a factor} \\ 1 : \text{answer} \end{cases}$

Question 2



A particle starts at point A on the positive x-axis at time $t = 0$ and travels along the curve from A to B to C to D, as shown above. The coordinates of the particle's position $\left(x(t),\, y(t)\right)$ are differentiable functions of t, where $x'(t) = \dfrac{dx}{dt} = -9\cos\left(\dfrac{\pi t}{6}\right)\sin\left(\dfrac{\pi\sqrt{t+1}}{2}\right)$ and $y'(t) = \dfrac{dy}{dt}$ is not explicitly given. At time $t = 9$, the particle reaches its final position at point D on the positive x-axis.

(a) At point C, is $\dfrac{dy}{dt}$ positive? At point C, is $\dfrac{dx}{dt}$ positive? Give a reason for each answer.

(b) The slope of the curve is undefined at point B. At what time t is the particle at point B?

(c) The line tangent to the curve at the point $\left(x(8),\, y(8)\right)$ has equation $y = \dfrac{5}{9}x - 2$. Find the velocity vector and the speed of the particle at this point.

(d) How far apart are points A and D, the initial and final positions, respectively, of the particle?

(a) At point C, $\dfrac{dy}{dt}$ is not positive because $y(t)$ is decreasing along the arc BD as t increases.

At point C, $\dfrac{dx}{dt}$ is not positive because $x(t)$ is decreasing along the arc BD as t increases.

$$2 : \begin{cases} 1 : \dfrac{dy}{dt} \text{ not positive with reason} \\ 1 : \dfrac{dx}{dt} \text{ not positive with reason} \end{cases}$$

(b) $\dfrac{dx}{dt} = 0;\ \cos\left(\dfrac{\pi t}{6}\right) = 0$ or $\sin\left(\dfrac{\pi\sqrt{t+1}}{2}\right) = 0$

$\dfrac{\pi t}{6} = \dfrac{\pi}{2}$ or $\dfrac{\pi\sqrt{t+1}}{2} = \pi;\ t = 3$ for both.

Particle is at point B at $t = 3$.

$$2 : \begin{cases} 1 : \text{sets } \dfrac{dx}{dt} = 0 \\ 1 : t = 3 \end{cases}$$

(c) $x'(8) = -9\cos\left(\dfrac{4\pi}{3}\right)\sin\left(\dfrac{3\pi}{2}\right) = -\dfrac{9}{2}$

$\dfrac{y'(8)}{x'(8)} = \dfrac{dy}{dx} = \dfrac{5}{9}$

$y'(8) = \dfrac{5}{9}x'(8) = -\dfrac{5}{2}$

The velocity vector is $< -4.5, -2.5 >$.

Speed $= \sqrt{4.5^2 + 2.5^2} = 5.147$ or 5.148

$$3 : \begin{cases} 1 : x'(8) \\ 1 : y'(8) \\ 1 : \text{speed} \end{cases}$$

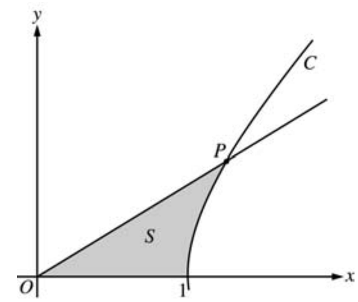
(d) $x(9) - x(0) = \displaystyle\int_0^9 x'(t)\,dt$

$= -39.255$

The initial and final positions are 39.255 apart.

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

Question 3



The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1+y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

(a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

(b) Set up and evaluate an integral expression with respect to y that gives the area of S.

(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

(a) At P, $\frac{5}{3}y = \sqrt{1+y^2}$, so $y = \frac{3}{4}$.

Since $x = \frac{5}{3}y$, $x = \frac{5}{4}$.

$$\frac{dx}{dy} = \frac{y}{\sqrt{1+y^2}} = \frac{y}{x}. \text{ At } P,\ \frac{dx}{dy} = \frac{3/4}{5/4} = \frac{3}{5}.$$

$2: \begin{cases} 1: \text{coordinates of } P \\ 1: \frac{dx}{dy} \text{ at } P \end{cases}$

(b) $$\text{Area} = \int_0^{3/4}\left(\sqrt{1+y^2} - \frac{5}{3}y\right)dy$$
$$= 0.346 \text{ or } 0.347$$

$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(c) $x = r\cos\theta\,;\ y = r\sin\theta$

$$x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$$

$2: \begin{cases} 1: \text{substitutes } x = r\cos\theta \text{ and} \\ \quad y = r\sin\theta \text{ into } x^2 - y^2 = 1 \\ 1: \text{isolates } r^2 \end{cases}$

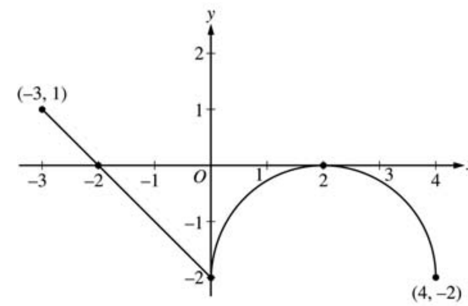
(d) Let β be the angle that segment OP makes with the x-axis. Then $\tan\beta = \frac{y}{x} = \frac{3/4}{5/4} = \frac{3}{5}$.

$$\text{Area} = \int_0^{\tan^{-1}(3/5)} \frac{1}{2}r^2\,d\theta$$
$$= \frac{1}{2}\int_0^{\tan^{-1}(3/5)} \frac{1}{\cos^2\theta - \sin^2\theta}\,d\theta$$

$2: \begin{cases} 1: \text{limits} \\ 1: \text{integrand and constant} \end{cases}$

Question 4

Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown above.



Graph of f'

(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

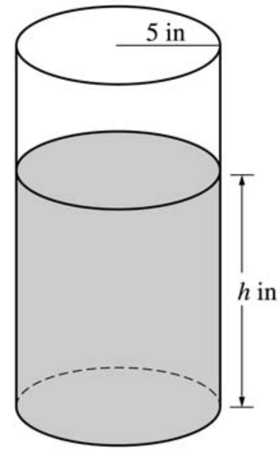
(c) Find an equation for the line tangent to the graph of f at the point $(0,3)$.

(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

(a) The function f is increasing on $[-3,-2]$ since $f' > 0$ for $-3 \le x < -2$.	$2 : \begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$
(b) $x = 0$ and $x = 2$ f' changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$	$2 : \begin{cases} 1 : x = 0 \text{ and } x = 2 \text{ only} \\ 1 : \text{justification} \end{cases}$
(c) $f'(0) = -2$ Tangent line is $y = -2x + 3$.	1 : equation
(d) $f(0) - f(-3) = \int_{-3}^{0} f'(t)\,dt = \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}$ $f(-3) = f(0) + \frac{3}{2} = \frac{9}{2}$ $f(4) - f(0) = \int_{0}^{4} f'(t)\,dt = -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi$ $f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$	4 : 1 : $\pm\left(\frac{1}{2} - 2\right)$ (difference of areas of triangles) 1 : answer for $f(-3)$ using FTC 1 : $\pm\left(8 - \frac{1}{2}(2)^2\pi\right)$ (area of rectangle – area of semicircle) 1 : answer for $f(4)$ using FTC

Question 5

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

$$3 : \begin{cases} 1 : \dfrac{dV}{dt} = -5\pi\sqrt{h} \\ 1 : \text{computes } \dfrac{dV}{dt} \\ 1 : \text{shows result} \end{cases}$$

(b) $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}}\,dh = -\frac{1}{5}\,dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

$$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } h = 17 \\ \quad\ \text{when } t = 0 \\ 1 : \text{solves for } h \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\dfrac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

1 : answer

Question 6

The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

(a) $f'(0) =$ coefficient of x term $= 0$

$f''(0) = 2$ (coefficient of x^2 term) $= 2\left(-\frac{1}{3!}\right) = -\frac{1}{3}$

f has a local maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

$4 : \begin{cases} 1 : f'(0) \\ 1 : f''(0) \\ 1 : \text{critical point answer} \\ 1 : \text{reason} \end{cases}$

(b) $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so $\left|f(1) - \left(1 - \frac{1}{3!}\right)\right| \le \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$.

$1 : \text{error bound} < \frac{1}{100}$

(c)

$$y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots + \frac{(-1)^n 2n x^{2n-1}}{(2n+1)!} + \cdots$$

$$xy' = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots + \frac{(-1)^n 2n x^{2n}}{(2n+1)!} + \cdots$$

$$\begin{aligned} xy' + y &= 1 - \left(\frac{2}{3!} + \frac{1}{3!}\right)x^2 + \left(\frac{4}{5!} + \frac{1}{5!}\right)x^4 - \left(\frac{6}{7!} + \frac{1}{7!}\right)x^6 + \cdots \\ &\quad + (-1)^n\left(\frac{2n}{(2n+1)!} + \frac{1}{(2n+1)!}\right)x^{2n} + \cdots \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots \\ &= \cos x \end{aligned}$$

$4 : \begin{cases} 1 : \text{series for } y' \\ 1 : \text{series for } xy' \\ 1 : \text{series for } xy' + y \\ 1 : \text{identifies series as } \cos x \end{cases}$

OR

$$xy = xf(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \cdots$$

$$= \sin x$$

$$xy' + y = (xy)' = (\sin x)' = \cos x$$

OR

$4 : \begin{cases} 1 : \text{series for } xf(x) \\ 1 : \text{identifies series as } \sin x \\ 1 : \text{handles } xy' + y \\ 1 : \text{makes connection} \end{cases}$