Student Performance Q&A:

2003 AP® Calculus AB & AP Calculus BC Free-Response Questions

The following comments on the 2003 free-response questions for AP® Calculus AB and AP Calculus BC were written by the Chief Reader, Larry Riddle of Agnes Scott College in Decatur, Georgia. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student performance in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

AB Question 1/BC Question 1

What was the intent of this question?

This problem presented a region bounded by two graphs and a vertical line. Students needed to use integration to find an area and two volumes. A graphing calculator was required to find the point of intersection of the two graphs. Part (a) required the use of a definite integral to find the area of the region. Part (b) required another application of integration to find the volume generated by revolving this region about a horizontal line. The resulting solid had cross sections in the shape of washers. Part (c) again required integration, this time to find the volume of a solid whose base was the region and whose cross sections were rectangles.

How well did students perform on this question?

Although this was the first free-response problem, and the problem was mostly quite traditional, the AB students did not seem to score as strongly as expected. Reasons might include the request to revolve about a line that was not an axis, or to find a volume of a different type of solid (rectangular cross sections) for which the description required careful reading. The mean score was 3.90 out of a possible nine points for the AB students and 5.81 out of a possible nine points for the BC students. Almost 11% of the AB students and 24% of the BC students received a 9. But 20.6% of the AB students earned no points at all on this problem, a discouraging figure for such a standard type of calculus problem.

What were common student errors or omissions?

Despite a clear picture illustrating the region of interest, many students wrote integrals using a lower limit of $x = 0$. Some students did Part (a) via antidifferentiation, but few did Part (b) that way, and Part (c) could not be computed by antidifferentiation using familiar functions (though that did not stop some students from attempting it).

The most common errors in Part (b) were to set up an integral expression for rotation about the x-axis and to take the square of the difference of the functions. The most common error in Part (c) was to believe that each rectangular cross section had a constant height of 5.

Some students wrote integrals in Part (b) for the solid obtained by revolving about the vertical line $x = 1$ rather than the horizontal line $y = 1$, and others wrote integrals in Part (c) for solids with rectangular cross sections perpendicular to the y-axis. Because of concern about the presence of the vertical line $x = 1$ in the picture in the stem, and the lengthy wording of the description of the solid in Part (c), the scoring allowed students in these two cases to earn one of the three possible points for a completely correct solution. This was a departure from the usual practice of not awarding points when students do not solve "our problem," and it may not necessarily be repeated in similar situations in the future.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Students did not need to compute antiderivatives to compute the definite integrals in Parts (a) and (b); the calculator could be used to calculate the value of the definite integral without further explanation once the setup of the definite integral is shown.

Students needed to find the point of intersection of the two graphs given in the stem. This should be done by solving the appropriate equation, not by tracing along the graphs. While the coordinates of the intersection are not technically a "final answer," they should still be reported to at least 3 decimal places when used in the limits of a definite integral to conform with the requirements for decimal answers as described in the instructions to the exam.

Students need as much practice as possible with setting up integrals for regions revolved about horizontal lines other than the x-axis.

Decimal answers with no supporting work received no credit on this problem, even if correct. It is important for students to show their work on the free-response section of the exam. Have them clearly show the setup for all definite integrals that are evaluated numerically. Consult the Web page "Free-Response Instruction Commentary" in the Teachers' Corner for Calculus AB or Calculus BC at AP Central™ (apcentral.collegeboard.com) for comments about the general instruction to "show all your work."

AB Question 2

What was the intent of this question?

This problem presented the velocity function for a particle moving along the x-axis and asked questions about acceleration, change of direction, distance traveled, and greatest distance from the origin — questions that involved interpretation of velocity and knowledge of its relationship to the position of the particle. Both differentiation at a point and integration over a specified interval were used in solving the problem.

Part (a) required students to differentiate the velocity function at a specific time in order to find the acceleration. Students needed to understand the difference between velocity and speed. One way to decide whether or not the speed was increasing was to recognize that positive acceleration and negative velocity imply decreasing speed. Part (b) required students to use the change in the sign of the velocity function to determine where the particle changes direction. Part (c) asked students to set up and compute a definite integral that gave the distance traveled over a time interval. This could be done either by using the turning point from Part (b) or by integrating the speed. Part (d) brought everything together, asking for the furthest distance from the origin. Because there was no closed formula for the position, students needed to know how to use the definite integral of the velocity to determine position. The answer to Part (b) provided one candidate for a time at which the particle could be furthest from the origin. It was necessary to find the position of the particle at this time. Students also needed to compare this distance to the distance from the origin at times $t = 0$ and $t = 3$.

How well did students perform on this question?

The mean score was 3.00 out of a possible nine points for the AB students. However, only 0.4% of the students received all nine points, while 17.7% earned no credit.

What were common student errors or omissions?

Students did reasonably well on those parts of the problem for which numerical answers could be obtained from the calculator. Most difficulties were in the justification portions of the problem, or in either forgetting or disregarding the initial position of the particle by the time the students got to Part (d). Students usually erred in Part (a) by believing the speed was increasing because the acceleration was positive. Virtually no student attempted to use the calculator to compute a numerical derivative of the speed function. In general, it is difficult for students to justify increasing/decreasing behavior by appealing to the behavior of a graph rather than by using calculus concepts.

Some students answered Part (b) by giving the number of times the particle changed direction rather than the time at which this happened. Other common errors in Part (b) were addressing points not within the specified domain of $0 < t < 3$ and not specifically presenting some argument about why the velocity changes sign.

Part (c) could be correctly done even if students computed an incorrect turning point or no turning point in Part (b). The most common error in Part (c) was not integrating the absolute value of the velocity function. In Part (d) the numerical answers were easy to evaluate, but the most common error was not using the initial condition and/or not reporting distance as a positive number.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Teachers should do more motion problems. Motion is a very typical application in the use of calculus and easy for students to understand physically. Experiments or labs might be done in the classroom so that students can get a tactile and graphical understanding of motion and the effects of a changing velocity, speed, and acceleration on objects in motion.

Readers were greatly relieved by the relatively small number of students whose calculators could be discerned to be in “degree mode.” Most degree-mode solvers rarely got past Part (a) and did not do well on the problem. Teachers are urged to remind their students to set their calculators to radian mode before entering the exam room.

Have students show their work. Clearly show the setup for all definite integrals that must be evaluated.

AB Question 3

What was the intent of this question?

This problem presented the rate of fuel consumption, $R(t)$, of an airplane. The rate was given in both graphical and tabular form. Students were asked questions that addressed the values and meanings of the derivative, the second derivative, and the integral of the function describing the rate of consumption. Units of measure were important in Parts (a) and (d). In Part (a) students were required to know how to use the values in the table to find an approximation to the derivative of *R* at time $t = 45$. Students should have used those values closest to $t = 45$. Part (b) required students to recognize that when the rate of increase of *R* is maximized, the second derivative of *R* will be zero. For Part (c) students needed to calculate a left Riemann sum approximation to a definite integral and to know that for an increasing function this always underestimates the true value of the integral. Part (d) asked for the meaning of the definite integral of *R*. It inverted the usual question about the average value of a function by presenting the definite integral divided by the length of the time interval and asking students for its meaning in terms of the fuel consumption of the airplane. Reference to the specific time interval was an important part of the explanation.

How well did students perform on this question?

The mean score was 2.42 out of a possible nine points, indicating a poor performance by the AB students. From a statistical point of view, this was the hardest problem on the AB exam. Only 1.1% of the AB students earned all nine points, while almost 30% of the students earned no credit.

What were common student errors or omissions?

Four of the nine points on this problem involved explanations. A significant number of students were unable to make an entry into the problem or only a minimal one even though similar reform concepts have been tested on recent AP Calculus AB Exams. Some students omitted the units from their answer in Part (a) or gave incorrect units (e.g., gallons per minute). In Part (b) many students said that $R''(45) = 0$ because the graph of *R* had a point of inflection at $t = 45$. This conclusion could not be justified from the given information, however, because *R* could have been linear on the interval $40 < t < 50$. In Part (c) students had problems calculating the value of the left Riemann sum since the subintervals given in the table were not of uniform length. Some students assumed a uniform length while others made careless errors in copying numbers from the table or doing simple computations. Finally, in Part (d) many students claimed that the second integral was the average value of fuel consumption rather than the average value of the rate of fuel consumption. They could still earn credit if the units clearly indicated that the average value applied to the rate.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

One of the primary goals of the changes that AP Calculus has undergone in the past several years is to move away from testing rote manipulation and toward problems that probe an understanding of the fundamental concepts. This problem was an example of one that required students to make connections between calculus and the “real” world, and between topics within calculus itself. Students need additional practice in working with functions that are given only in numerical or tabular form and in using appropriate mathematical terminology when explaining the meaning of mathematical expressions. They also need to have a very good understanding of rates of change in order to give correct interpretations and units. Finally, students should see more examples of calculating Riemann sums with subintervals of different lengths.

AB Question 4/BC Question 4

What was the intent of this question?

This problem presented students with the graph of the derivative of f. It explored their understanding of the relationship between the behavior of the derivative and of the function f. In Part (a) students needed to know how to use the sign of the derivative to determine when the function is increasing. Part (b) could be answered by knowing how to determine the concavity of f by observing where the graph of f' is increasing or decreasing. Part (c) required knowing the relationship between the values of f' and the slope of the tangent line of f. Part (d) required an application of the Fundamental Theorem of Calculus, recognizing that $f(b)-f(a)$ is the integral of f' over the closed interval $[a, b]$, and then using this and the fact that $f(0)=3$ to calculate the values of f at $x=-3$ and $x=4$. The value of f at $x=4$ is most easily computed by finding the area between the graph of f' and the x-axis (a rectangle minus a semicircle) and subtracting this area from the value of f at $x=0$.

How well did students perform on this question?

The mean score was 2.68 out of a possible nine points for the AB students and 4.13 out of a possible nine points for the BC students. About 20.4% of the AB population and 6.5% of the BC population earned no points on this problem, a disappointment considering the emphasis placed on analyzing the behavior of a function given the graph of its derivative on previous AP Calculus Exams.

What were common student errors or omissions?

Most students felt comfortable enough to attempt Part (a). Many students who may have, in fact, understood the problem lost points because they used ordered pairs from the graph of f' instead of presenting an interval for the answer to Part (a), or they lost the reason point because of ambiguity about whether the explanation applied to the function f' or f, or they made generic statements about f rather than referring to the specific behavior of the graph of f'.

In Part (b) students were clearly comfortable with points of inflection where $f''(x)=0$ such as occurred at $x = 2$, but a significant number of students were confused about how to handle the inflection point at $x = 0$ where the second derivative does not exist. A large number of students identified $x = -2$ as a candidate, whether by misreading the coordinates of the inflection point at (0, –2) or by misreading the graph of f' and identifying the critical point of f.

Most students earned the point for the tangent line in Part (c). The AB students, however, found Part (d) to be extremely difficult. Of those who recognized the area connection, many could calculate the areas correctly but either made sign errors or failed to recognize the need to use the Fundamental Theorem of Calculus. A small percentage of students attempted to solve Part (d) analytically. Few were successful.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

In order that their students might perform better on this sort of problem, teachers are urged to continue to emphasize the Fundamental Theorem of Calculus as well as a graphical approach to problem solving. Students continue to have difficulty with properly justifying their conclusions, and more experience with this would improve their performance on the exam. Students need to practice mathematical writing skills to help communicate their reasoning and explanations to the readers. This has become an increasingly important part of the free-response problems. Functions should be referred to by the name used in the stem of the problem, not using a generic "it."

Teachers may want to provide students with more practice problems where inflection points occur at places where the value of the second derivative does not exist.

AB Question 5/BC Question 5

What was the intent of this question?

This problem presented a related rates setting as well as a separable differential equation. The volume of coffee in a cylindrical pot was changing at a rate that was given as a function of the height. Part (a) asked students to find the rate of change of height as a function of the height. This required translating the information on the rate of change of the volume V into mathematical notation and showing how to use the chain rule to find the desired result. The solution was given so that all students could start Part (b) on an equal footing. Part (b) asked students to solve the separable differential equation that was the solution to Part (a). Part (c) used the answer to Part (b) and required recognition that the coffeepot was empty precisely when the height was zero.

How well did students perform on this question?

The mean score was 3.26 out of a possible nine points for the AB students and 5.88 out of a possible nine points for the BC students. About 6% of the AB students earned all nine points, perhaps not too bad for a problem that involved both a related rates question and solving a separable differential equation. Moreover, while almost 23% of the AB population could not earn any points on this problem, that is still an improvement from the last separable differential equation problem in 2001 when 30% earned no points. About 21% of the BC students earned all nine points, with 6% earning no points.

What were common student errors or omissions?

Most students correctly translated the given rate information in the stem of the problem into a mathematical expression for $\frac{dV}{dt}$. When computing $\frac{dV}{dt}$, however, many students correctly treated h as a variable but made mistakes in differentiation. Many students used prime notation, differentials, or delta notation to express their rates of change with respect to time, and these were accepted if the meaning was not ambiguous. Notation such as $V'(h)$, however, was *not* accepted for a rate of change with respect to time.

The standard for Part (b) represented what has become the usual approach to grading separable differential equation problems. The first two points were for mechanical operations of separating the variables and antidifferentiating the two sides. The last three points were for finding the particular solution satisfying the given initial condition. Inclusion of the constant of integration was critical because without it students could not proceed from a general solution to the particular solution. Common errors included incorrect separations of the variables, incorrect antidifferentiation, and algebra errors in solving for h in terms of t.

Answers to Part (c) were accepted if consistent with an incorrect result from Part (b), unless students reported a negative value of t as the time when the coffeepot was empty.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

A significant number of students had difficulty correctly separating the variables for the differential equation. This continues to be a topic that requires additional practice. The antiderivatives that needed to be computed were relatively simple ones following directly from basic integration rules. Teachers may also want to remind their students to pay particular attention to how their answers should be presented (e.g., h as a function of t in this problem).

AB Question 6

What was the intent of this question?

This problem presented two piecewise-defined functions. It explored students' understanding of continuity, differentiability, and average value of a function. Part (a) required students to know that a function is continuous if and only if the value of the function agrees with both its limit from the left and its limit from the right. Part (b) was an average value problem. Students needed to know how to integrate a piecewise-defined function. Part (c) gave a piecewise-defined function with two free parameters which needed to be determined using the knowledge that the function was differentiable and hence also continuous at $x = 3$. These conditions produced two linear equations in the two unknowns.

How well did students perform on this question?

This was the first time a problem of this type involving continuity and differentiability had appeared on the AB free-response section in about 20 years. Nevertheless, many students did attempt this problem, as reports indicated that relatively few students left the problem blank (only about 3% received a dash indicating no mathematical work on the page). The mean score, however, was only 2.68 out of a possible nine points. It was very difficult to earn all nine points; only 0.7% of the students achieved this, and almost 52% of the students earned at the most two points.

What were common student errors or omissions?

Students could earn the first point in Part (a) with the correct answer and equating the values of the left- and right-hand limits. However, they needed to provide a more complete explanation involving the use of limits to earn the second point.

In Part (b) a number of students tried to split the integration and do an “average value” on each part separately, using a constant of 1/3 with the first integral and a constant of 1/2 with the second. These students could not earn the first or fourth points but were eligible for both antiderivative points.

Students did not have to provide reasons or explanations in Part (c). If a function g is differentiable on an open interval containing $x = a$, and if both $\lim_{x \to a^-} g'(x)$ and $\lim_{x \to a^+} g'(x)$ exist, then the two limits are equal and the common value is $g'(a)$. (This is a consequence of the Mean Value Theorem. For related ideas, see the article “Things I Have Learned at the AP Reading” by Dan Kennedy, *College Mathematics Journal* [November 1999]: 346.)

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Theory is still an important ingredient in the understanding of calculus and needs to be integrated into problem-solving strategies. Students need to practice mathematical writing skills to help communicate their reasoning and explanations to the readers.

BC Question 2

What was the intent of this question?

This problem dealt with particle motion in the plane. It presented an explicit formula for $x'(t)$ only. A graph of the particle’s motion was provided, giving some information on the slope of the tangent to the path of the particle at four points. Part (a) asked for the signs of $x'(t)$ and $y'(t)$ at one of these points. One approach was to use the relationship between the signs of x' and y' and the sign of the slope of the tangent line. A simpler approach was to recognize that both the x- and y-coordinates of the point are decreasing as the particle moves along the path through the point C. The sign of $x'(t)$ could also be read from the derivative. Part (b) required students to recognize that if x and y are differentiable functions of t, then the slope of the tangent can be undefined only when $x'(t) = 0$. Students then needed to solve this equation. In Part (c) students were required to know the relationship connecting $x'(t)$, $y'(t)$, and the slope of the tangent line. Furthermore, they needed to know how to use these values to find the velocity vector and the speed. Part (d) asked for the total change in the x-coordinate, requiring students to set up and evaluate a definite integral.

How well did students perform on this question?

The mean score was 4.11 out of a possible nine points. Almost 7% of the students earned all nine points, while about 12.3% earned no points.

What were common student errors or omissions?

In Part (a) students had to connect the downward motion to the sign of $\frac{dy}{dt}$ and the leftward motion to the sign of $\frac{dx}{dt}$. The wide variation in language that students used in Part (a) was the most difficult aspect of scoring this problem. Subtle nuances of language had to be interpreted to distinguish between valid and invalid reasoning.

Students were generally successful on Part (b). Some students found additional times outside the given domain, and these needed to be eliminated to gain full credit. In Part (c) recognizing the role of 5/9 in relating the derivatives of $x(t)$ and $y(t)$ at $t = 8$ was viewed as fundamental to the problem. Most errors were either in failing to make this connection or making arithmetic mistakes in the computation. Students did not always make the connection between the velocity vector and speed.

Attempts to use a general arc length formula using both $\frac{dx}{dt}$ and $\frac{dy}{dt}$, or using $\frac{dy}{dx}$, were the most common mistakes in Part (d).

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Teachers are encouraged to reinforce the difference between vector values (such as a velocity vector) and scalar values (such as speed).

BC Question 3

What was the intent of this question?

This problem gave a region bounded by two graphs and the x-axis. Students needed to show knowledge of how to calculate this area using both rectangular and polar coordinates. Part (a) required students to find the coordinates of the point of intersection of the two graphs and to find the value of the derivative of x with respect to y on one of the curves at that point. In Part (b) students were required to find the area by setting up and evaluating a definite integral with respect to y. Part (c) required students to demonstrate knowledge of how to transform an equation in rectangular coordinates into one in polar coordinates. Part (d) required students to know how to set up an integral in polar coordinates for the area of the region. While it was not necessary to evaluate this integral, such an evaluation, compared with the answer to Part (b), would have provided a check on the answers to Parts (b) and (d).

How well did students perform on this question?

This question was split scored, with Parts (a) and (b) contributing to the Calculus AB subscore grade and parts (c) and (d) being BC-only material (polar coordinates and polar area.) The mean score on Parts (a) and (b) was 3.33 (out of a possible five points), with about 26% earning all five points. The mean score on Parts (c) and (d) was 1.66 (out of a possible four points), with 11% earning all four points but 33.7% earning none of the four points. The total mean score was 4.99 out of a possible nine points.

What were common student errors or omissions?

Most students used their calculators to solve for the y component in Part (a), but many computed $\frac{dx}{dy}$ without using the calculator. The most common error was to compute $\frac{dy}{dx}$ rather than $\frac{dx}{dy}$, or to either not evaluate $\frac{dx}{dy}$ or evaluate it incorrectly. Most students were able to set up an integral in Part (b), though some attempted to find the area by using vertical slices despite the explicit instructions to do an integral with respect to the y variable. Part (c) was relatively easy for students, but many were unable to find the upper limit of integration in Part (d), wrote only an indefinite integral, or showed only a generic polar area integral formula without relating it to this region or polar function.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Have students show their work. It is important to write the explicit equation that is being solved on a calculator, or the specific function whose derivative is computed numerically with the calculator.

Polar coordinates and polar area are still important parts of the BC course. Two parts of this problem asked students to set up an integral expression with respect to a specific variable. Students need to experience a variety of situations in which integrals can be written in terms of variables other than the usual x variable.

BC Question 6

What was the intent of this question?

This problem involved a power series. Students were given the power series expansion of a function and then asked questions that probed their understanding of this function. Part (a) required that students know how to read or compute the values of the first and second derivative at $x = 0$ from the series. Students then needed to use this information to describe the nature of the critical point at $x = 0$. In Part (b) students were required to know that with an alternating series whose terms are decreasing and approaching zero in absolute value, the error introduced by truncating the series has absolute value bounded by the absolute value of the next term. Part (c) presented a differential equation and asked students to show that the given power series satisfied that equation. Students could do this by differentiating the power series, multiplying it by x, adding the original power series to this product, and then recognizing the result as the power series expansion for cosine. Alternatively, this part could be done by recognizing that x times the power series is the series for sine, the derivative of xy is $xy' + y$, and the derivative of sine is cosine.

How well did students perform on this question?

All BC exams in recent years have had a problem with series, and student performances have generally been weak. But this year the mean score on the series question was 2.66 out of a possible nine points, which is the lowest series mean since 1997. Only 2.5% earned all nine points, and about 25.7% earned no points.

What were common student errors or omissions?

In Part (a) many students were unable to correctly compute the first and/or second derivative at $x = 0$. Disappointingly, many tried by actually taking the derivative of the series rather than using the relationship between the derivatives at $x = 0$ and the coefficients of the terms of the series. Perhaps the most common error in Part (a) was failing to use both the first and second derivatives in providing a reason for the local behavior at $x = 0$. It was extremely difficult to justify a reason based on a first derivative test.

In Part (b) the three most common results were to get the one available point, to have no discernible answer on the page, or to fail to earn the point by claiming the error to be equal to 1/120 rather than bounded by 1/120.

Student errors in Part (c) were numerous and varied. The largest proportion of students attempted Part (c) by a variation of the first method on the standard. Common errors were not dealing with an infinite series or not using enough nonzero terms. Also, many students did not show enough details in the gathering of like terms to convince readers that they had arrived at the (already known) destination of $\cos x$ on the right-hand side of the differential equation. By far, the most common point lost in Part (c) by those with substantive answers not of the form of the first method on the standard was due to the introduction of x in the denominator without properly handling the case when $x = 0$.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

Students need additional practice manipulating the general term of a series and recognizing the difference between working with Taylor polynomials and infinite series.

No justification was required this year for the work on Part (b). Many students said that because the series was alternating, the error for the truncated sum was less than the first omitted term in the series. Teachers need to remind students that the error estimate depends not only on having an alternating series but also on having terms that decrease to 0 in absolute value. Students may lose points on future exams if they do not give or verify all necessary conditions in such situations.

Reminder: Consult the Web page “Free-Response Instruction Commentary” in the Teachers’ Corner for Calculus AB or Calculus BC at AP Central (apcentral.collegeboard.com) for additional comments about the general instruction to “show all your work.”