One Hundred and Fifty Years of Teaching Calculus

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Suppose you wanted to know what a calculus course was like 50 or 100 or even 150 years ago. Where would you turn? What would you study? There are student descriptions of what their classes and teachers were like; we even have some notebooks. But by far the most readily accessible source of information on what was covered and how it was covered comes from textbooks.

Textbooks mold our courses. We may not follow them exactly; we are likely to add and subtract, omitting a section here, adding interesting asides there. By and large, however, our courses are organized around our books. So looking at texts seems like a reasonable way to discover how calculus has changed, at least for the beginner.

I'd like to be able to tell you that I began studying calculus texts some 20 years ago because this was the question that fascinated me. But that's not the case. I began because I happened to have a copy of Elias Loomis's book of 1860, *Analytical Geometry and Calculus*[1]. What fascinated me was that Loomis appeared to know nothing of Cauchy's work on the foundations of calculus, written some 40 years earlier[2]. While I was clever enough to know that it took time for ideas to filter down, particularly across an ocean, 40 years seemed a long time. What I was to learn is that forty years is not so long and that more of Cauchy than I realized was in Loomis.

What I'd like to do is talk about several topics that are handled quite differently, here at the beginning of the twenty-first century, than they were in the nineteenth, hoping you'll find something novel to take back to your classroom. As an historian, I want to be fair to the materials I present by making it clear that these are not silly men who did not know any better, but conscientious teachers trying to do right by their not always enthusiastic students.

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[1] Loomis, Elias: *Elements of analytical geometry and of the differential and integral calculus*, New York, Harper and Brothers, 1860
[2] Cauchy, A. L., *Cours d'analyse de l'école royal polytechnique. 1[re] partie: analyse algébrique*, Paris, 1821.

NOTATION

Let's start with notation. When we write " $\lim_{x \to a} f(x) = L$ ", we have made use of a very powerful notation. First, we have a strong intuitive notion of what this means: "As *x* gets close to a, $f(x)$ gets close to *L*." Or, perhaps: "We can make *f*(*x*) as close to *L* as we like by simply requiring that *x* be close (but not equal) to
a." Or, in utter desperation, we might even be driven to say: "Given $\varepsilon > 0$, there is a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$." We are even naïve enough to believe that this last formulation is, somehow, "natural." In fact, I came to the realization that most math majors do not understand this "natural" formulation and how it is to be used while (unsuccessfully) teaching Real Analysis for the twentieth or thirtieth time – I'm a slow learner – and that it was no wonder that students had trouble with the concept of limit in Calculus 1!

Now imagine that you were teaching calculus in 1840. You are using the very popular text by Charles Davies,[3] and you want to teach your class how to find derivatives. You need to understand that during this period, if calculus was being taught, it was being taught to all students. The curriculum had no options. Furthermore, students generally came to college with only a smattering of algebra and geometry, totally unprepared to face calculus until much later. At Harvard in 1830, for example, sophomores studied trigonometry and its applications, topography, and calculus; this third of a year was all of the calculus available to them.[4] To make matters worse, you have neither limit notation nor firm functional ideas available to you.

How are you going to be able to do this? Well, here's what Davies did:

Davies begins by noting that "if two variable quantity are so connected to each other that any change in the value of one necessarily produces a change in the value of the other, they are said to be functions of each other."[5] This symmetric view of the functional relationship will prove very handy, for the text emphasizes the calculus of curves, as opposed to that of a function. (In practice, Davies is going to work with analytic expressions like "$x^2 + y^2 = 2ax$". While he introduces the notation "$y = f(x)$"[6], he seldom uses it.)

[3] Davies, Charles, *Elements of the differential and integral calculus*, improved edition, New York, A. S. Barnes & Co., 1836.
[4] *Ibid*, p. 132.
[5] Davies, *op. cit.*, p. 9.
[6] Davies, *op. cit.*, p. 10.

By page 15, Davies is ready to tackle what happens when the independent variable in a functional relationship is incremented by *h*. As a good teacher should, he first looks at a couple of examples, $u = ax^2$ and $u = x^3$. Letting *u*' be the incremented value of the function, he looks at $\frac{u' - u}{h}$ and declares

> If we examine the second members of these equations, we find a term in each which does not contain the increment *h*. … If now, we suppose *h* to diminish, it is evident that the terms $2ax$ and $3x^2$, which do not contain *h*, will remain unchanged, while all the terms which contain *h* will diminish. Hence the ratio
>
> $$\frac{u' - u}{h}$$
>
> in either equation will change with *h*, so long as *h* remains in the second number of the equation; but of all the ratios which can subsist between
>
> $$\frac{u' - u}{h}$$
>
> is there one which does not depend on the value of *h*? We have seen that as *h* diminishes, the ratio in the first equation approaches $2ax$, and in the second to $3x^2$; hence $2ax$ and $3x^2$ are the limits toward which the ratios approach in proportion a[s] *h* is diminished; and hence each expresses that particular ratio which is independent of the value of *h*. This ratio is called the limiting ratio of the increment of the variable to the corresponding increment of the function. (Pp. 17, 18)

Davies goes on to say that "the limiting ratio of the increment of the variable to that of the function … is called the differential coefficient of *u* regarded as a function of *x*." (p.19) Indeed, he never uses the term "derivative," but sticks with "differential coefficient," a term introduced by Lagrange.[7] He now immediately jumps to using differentials and infinitesimals by saying "represent by *dx* the last value of *h*, that is the value of *h*, which cannot be diminished according to the law of change to which *h* or *x* is subjected, without becoming 0 and let us also represent by *du* the corresponding difference between *u'* and *u*; …"(p. 18)

[7] Lagrange, J. L., *Théorie des functions analytiques*, Paris, 1797.

Notice that Davies is in a very awkward position: He has no notation for calculating derivatives, so he must go back to first principles. He gets around this by introducing a property of the derivative used by Lagrange,[8] namely that

$$u' - u = Ph + P'h^2$$

where *P* is the differential coefficient and *P'* will in general be a function of *h* as well as *x*. He explicitly assumes this on the basis of his previous examples. (p. 21) Now he's in business! He can use this to derive the rules of differentiation. For example, you would find his proof of the product rule quite familiar, although perhaps not the form of the result. Please remember that when Davies writes "u'", he does not mean the derivative of u, but $u' = f(a+h)$ in our notation:

Suppose $u' - u = Ph + P'h^2$ and $v' - v = Qh + Q'h^2$.
Then $(uv)' - uv = u'v' - uv = u'(v'-v) + (u'-u)v = u'(Qh + Q'h^2) + (Ph + P'h^2)v.$
Or $(uv)' - uv = u'v' - uv = u'(v'-v) + u'-u)v = u'(Qh + Q'h^2) + (Ph + P'h^2)v.$

Remembering that $h = dx$, $\frac{(uv)' - uv}{h} = u'(Q + Q'h) + (P + P'h)v.$

So, "passing to the limit", as Davies would say, $\frac{d(uv)}{dx} = uQ + Pv = u\frac{dv}{dx} + \frac{du}{dx}v.$

Now multiply through by $\frac{dx}{uv}$: $d(uv) = \frac{du}{u} + \frac{dv}{v}$.

I need to say something here about notation and names for the derivative. You, of course, are very aware that we have several that are in common use, as well as the fact that there are several others that have either disappeared or are uncommon.

First, there are the notations of the founders, Leibniz's "$\frac{dy}{dx}$" and Newton's dots, where $\dot{x} = \frac{dx}{dt}$. The former is still very much with us, although we see that Davies, and most other nineteenth century writers treat it as a quotient; the latter is mostly gone, except for some physicist-types. Then there's the operator notation, $D_x f$, which appeared sometime in the twentieth century. Finally, there's the notation due to Lagrange, $f'(x)$, which we know and love.

[8] See Grabiner, Judith V., *The origins of Cauchy's rigorous calculus*, Cambridge, MA, MIT Press, 1981, p. 118ff, for a discussion of the importance of this result.

Lagrange wrote a very influential book in 1797, in which he attempted to build the calculus on a foundation of Taylor series.[9] His notion was this: If you develop $f(x+h)$ as a series in powers of h, say, $f(x+h) = f(x) + A_1 h + A_2 h^2 + A_3 h^3 + \cdots$, then $A_1 = f'(x)$, the first derived function of f, $2A_2 = f''(x)$, the second derived function of f, $6A_3 = f'''(x)$, the third derived function of f, and in general, $n!A_n = f^{(n)}(x)$, the n^{th} derived function of f.

Now, once you start thinking about this, you'll see lots of problems, the most obvious of which is "how do you find the series without being able to find derivatives?" since this is "clearly" Taylor's Theorem. The important answer is that lots of series, including those for trigonometric functions and logarithmic functions, were already well known, having been derived in very clever ways. (Lagrange is good reading, even in French, and I'd recommend it if you have a student who is into that kind of stuff.)[10] Most of Lagrange has disappeared from our courses, but the name, differential coefficient, shortened to derivative, and the prime notation, remain.

> Now look again at Davies's preferred product rule: "$d(uv) = \frac{du}{u} + \frac{dv}{v}$." While there are problems with this (suppose $u = 0$), it is very convenient for the extended product rule, often called Leibniz's Rule: What is the derivative of $u_1 u_2 u_3 \cdots u_n$? In this formulation, the answer is "obvious":
> $d(u_1 u_2 u_3 \cdots u_n) = \frac{du_1}{u_1} + \frac{du_2}{u_2} + \frac{du_3}{u_3} + \cdots + \frac{du_n}{u_n}$.

Before I look at Davies's notion of an integral, let me go back to look at Davies and his book a little more closely. Davies was born in 1798 in northern New York state. He was appointed to the United States Military Academy in 1814 and graduated in 1815, somewhat speedier than usual, but the so-called curriculum was not much at this time. However, he really did not leave the Academy until 1837, by which time he was Professor of Mathematics. He served in several other academic posts, including filling in for a year for Elias Loomis, and finished his academic career at Columbia University in 1865.[11] But Davies is best known as a writer of textbooks.

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[9] Lagrange, J. L., *Théorie des fonctions analytiques*, Paris, 1797.
[10] On Lagrange, and his development of derived functions, see any standard history of calculus, such as Boyer, Carl B., *The history of the calculus and its conceptual development*, New York, Dover Books, 1949.
[11] See Ackerberg-Hastings, Amy, "Charles Davies, Mathematical Businessman" in History of undergraduate mathematics in America, edited by Amy Shell-Gellasch, West Point, NY, 2001.

Davies wrote, or translated, or possibly even stole, books from arithmetic through calculus with surveying and descriptive geometry thrown in. Cajori, in his survey of mathematics teaching in 1890, said, 13 years after Davies' death, that Davies was "one whose name is known to nearly every schoolboy in our land,"[12] so popular were his texts.

Calculus books by Davies appeared between 1836 and 1901, a span of 65 years. We'll look at the 1843 edition of *Elements of the Differential and Integral Calculus*.[13]
In his preface, Davies says,

> The Differential and Integral Calculus is justly considered the most difficult branch of pure Mathematics.
>
> The methods of investigation are, in general, not as obvious nor the connection between the reasoning and the results so clear and as striking, as in Geometry, or the elementary branches of analysis.
>
> It has been the intention, however, to render the subject as plain as the nature of it would admit, but still, it cannot be mastered without patient and severe study.
>
> This work is what its title imports, an Elementary Treatise on the Differential and Integral Calculus. It might have been much enlarged, but being intended for a text-book, it was not thought best to extend it beyond its present limits. … [14]

Davies goes on to say that "the works of Bourcharlat and Lacroix have been freely used, although the general method of arranging the subjects is quite different from that adopted by either of those distinguished authors."[15] In this he is dissembling, because the book is really quite similar to Bourcharlet's[16], both in the text itself and in the examples used. Unfortunately, in some of the places at which he differed from Bourcharlet, he made bad choices.

Davies' book is broken into eight numbered chapters and a ninth chapter on integral calculus. Each chapter in turn is broken into short sections. The entire

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[12] Cajori, Florian, *The teaching and history of mathematics in the United States*, Washington, DC, U.S. Government Printing Office, 1890.
[13] Davies, Charles, *Elements of the differential and integral calculus*, improved edition, New York, A. S. Barnes & Co., 1836.
[14] *Ibid*. p. 3.
[15] *Ibid*. p. 4.
[16] Bourcharlat, J.-L., *Élémens de calcul différentiel et de calcul integral*, quatrième edition, considérablement augmentée, Paris, Bachelier …, 1830. Editions of this book appeared starting in 1813. It was translated into English in 1828.

book is 283 pages! My most recent copy of Thomas[17] is over 1000 pages. You would immediately notice in Davies that there are very few exercises or problems for the student to complete, a situation that would not really change until the twentieth century dawned.

I	Definitions and Introductory Remarks	VI	Application of the Differential Calculus to the Theory of Curves
II	Differentiation of Algebraic Functions – Successive Differentials – Taylor's and Maclaurin's Theorems	VII	Of Osculatory Curves – Of Evolutes
III	Of Transcendental Functions	VIII	Of Transcendental Curves – Of Tangent Planes and Normal Lines to Surfaces
IV	Development of Any Function of Two Variables – Differential of a Function of any number of Variables – Implicit Functions – Differential Equation of Curves – Of Vanishing Fractions		Integral Calculus
V	Of the Maxima and Minima of a Function of a Single Variable		

The only application of the derivative ever considered by Davies is curve sketching, which he does in detail, taking as many pages as he spends on all of the theory of the derivative including Taylor series.[18] He discusses not only maxima and minima, but cusps, multiple points, evolutes and involutes, osculating curves, and transcendental curves, including the cycloid. Having spent 188 pages on topics that include some we might teach in Calculus 3, he's ready to tackle integration.

For Davies, the integral is the antiderivative, "the method of finding the function which corresponds to a given differential."(p. 189) He goes on to explain on the next page that the integral symbol denotes a sum and "was employed by those who first used the differential and integral calculus, and who regarded the integral of $x^m dx$ as the sum of all products which arise by multiplying the m[th] power of *x* for all values of *x* by the constant *dx*." Davies now spends 50 pages on techniques of integration. On p. 252, he finally gets around to introducing the

[17] Finney, Ross, Maurice Weir and Frank Giordano, *Thomas' calculus*, 10th edition, Boston, Addison-Wesley, 2001
[18] *Rosenstein*, op. cit., p. 83.

notation "$\frac{1}{p}\int_0^b y^2 dy = \frac{b^3}{3p}$." The notion of a Fundamental Theorem of calculus is entirely missing! The book ends with 40 pages of geometric applications of the integral, including arc length (he already knows that $ds = \sqrt{dx^2 + dy^2}$), areas, and volumes including double integrals.[19]

We have seen here a very different view of the standard calculus course, and we should realize that, although the subject was about 150 years old at the time, it had a long way to go before it resembled the AP syllabus.

Of the eight authors who began publication before 1860, only one (Loomis) had studied abroad and four were products of West Point. Six of the eight used limits as their fundamental approach to the derivative, but two, Benjamin Peirce and William Smyth used infinitesimals.

(Peirce was the only "professional mathematician" in the list. He was a founding member of the National Academy of Sciences and the author of *Linear Associative Algebra*, America's first major contribution to pure mathematics. His calculus book, however, was a pedagogical nightmare.)

TABLE 1: Commercial Authors Who Began Publication Before 1860

Name	**Dates**	**Edit**	**Education**	**Positions**
Davies	1836-1868	many	USMA*	USMA, Columbia
Peirce	1841-1862	3	Harvard	Harvard
Church	1842-1872	many	USMA	USMA
M'Cartney	1844-1848	2	Jefferson	Lafayette
Loomis	1851-1902	many	Yale, Paris	Yale, Western Reserve, U. City of New York
Smyth	1854-1859	2	Bowdoin	Bowdoin
Courtenay**	1855-1876	8	USMA	USMA, U. PA, U. VA
Quinby	1856-1879	6	USMA	USMA, U. Rochester

Dates = span of frequent publication
Edit = number of editions
* United States Military Academy
** Courtenay died in 1853, leaving a manuscript.

[19] Much of what I've said above was blissfully stolen from my own paper referenced above.

INFINITESIMALS

Let me go on now to a different author, a different time period and a different kind of calculus book. After the Civil War, higher education was a booming business in America. The Morrill Act, creating the land-grant colleges, encouraged states to begin universities; the westward expansion provided people for these new universities; and a desire for a practical education, primarily engineering and agriculture, provided a more receptive audience for calculus courses. Furthermore, electives were becoming fashionable, so that the audience for calculus courses became more specialized. Indeed, the book we'll look at was written for the "purely optional course" at Columbia.[20] Finally, wealthy industrialists were funding universities, like Cornell and Johns Hopkins, both named for their benefactors. In the last quarter of the nineteenth century, German-style universities began to replace the classical college as the standard for higher education in America.[21]

[20] Peck, William Guy, *Practical treatise on the differential and integral calculus*, New York, A. S. Barnes &Co., 1870. All references are to this edition.

[21] See Rudolph, Frederick, *The American College and University: A History*, New York, 1968

TABLE 2: Commercial Authors Who Began Publication 1870-1895

Name	**Dates**	**Edit**	**Education**	**Positions**
Olney	1870-1885	4	no formal	Kalamazoo, U. Mich
Peck	1870-1877	5**	USMA*	USMA, U. Mich, Columbia
Johnson##	1873-1909	many	Yale	USNA#, Kenyon, St. John's
Buckingham	1875-1885	3	USMA	Kenyon
Byerly	1879-1902	many	Ph.D. Harvard	Cornell, Harvard
Bowser	1880-1907	many	Santa Clara, Rutgers	Rutgers
Osborne	1889-1910	many	Harvard	USNA#, MIT
Taylor	1884-1902	9	Colgate	Colgate
Bass	1887-1905	6	USMA	USMA*
Newcomb	1887-1889	2	Harvard	Naval Observatory, Nautical Almanac, Johns Hopkins U.

Dates = Span of frequent publication
Edit = number of editions
* United States Military Academy
# United States Naval Academy
** includes an edition published after 1877
includes the books written jointly with John Minot Rice
Authors in *italics* used infinitesimals; in **boldface**, rates

Looking at the list in Table 2, you will not see many familiar names. Two, however, should jump out at you. One is Byerly, the first Ph.D. on our list and one of Harvard's first as well. The second is Simon Newcomb, who was the fourth president of the AMS. In the "official" history of the AMS, his biography takes up more pages than the sum of those of the first three presidents. He was an internationally recognized astronomer.[22] The author I want to look at, however, is William Guy Peck.

Peck, born in 1820, was one of the "West Point boys," graduating first in his class in 1844 and returning to be assistant professor of natural and experimental philosophy and mathematics from 1846 to 1855. He was also Charles Davies

[22] Archibald, Raymond Clare, *A semicentennial history of the American Mathematical Society 1888 – 1938*, AMS, 1938.

son-in-law! After leaving the military academy, he taught physics and engineering at the University of Michigan for two years, before returning east to serve at Columbia from 1857 until his death in 1892. He was a very popular teacher and the author of a number of texts in both mathematics and science. We'll look at his 1870 book, *Practical Treatise on the Differential and Integral Calculus*.[23]

Peck based his calculus on infinitesimals, a technique that was still dying, as we shall see, 80 years later. Authors chose to use infinitesimals because they were simpler and they facilitated the application of calculus to practical problems. These authors knew about limits; they just thought they got in the way of learning calculus. Amazingly, I agree with them. The more I look at the use of infinitesimals, the better I like them! Unfortunately, the pure mathematician in me won't let go.

PART I –DIFFERENTIAL CALCULUS	III. Singular Points of Curves	II. Areas of Plane Curves
I. Definitions and Introductory Remarks	IV. Maxima and Minima	III. Areas of Surfaces of Revolution
II. Differentiation of Algebraic Functions	V. Singular Values of Functions	IV. Volumes of Surfaces of Revolution
III. Differentiation of Transcendental Functions	VI. Elements of Geometrical Magnitudes	PART V – APPLICATIONS TO MECHANICS AND ASTRONOMY
IV. Successive Differentiation and Development of Functions	VII. Application to Polar Coordinates	I. Centre of Gravity
V. Differentiation of Functions of Two Variables and of Implicit Functions	VIII. Transcendental Curves	II. Moment of Inertia
PART II – APPLICATIONS OF THE DIFFERENTIAL CALCULUS	**PART III – INTEGRAL CALCULUS**	III. Motion of a Material Point
I. Tangents and Asymptotes	PART IV – APPLICATIONS OF THE INTEGRAL CALCULUS	
II. Curvature	I. Lengths of Plane Curves	

[23] Peck, *op. cit*.

Peck begins by defining a function. He supposes that a relationship between two variables is expressed by an equation. Then one of the variables is called the independent variable and the other is said to be a function of the first. He also admits functions of several variables and uses functional notation. Now he wants to develop infinitesimals.

> A quantity is … *infinitely small* with respect to [another] when the quotient is less than any assignable number. If the term of comparison is finite, [these quantities are called] *infinitesimals*.
>
> Infinites and infinitesimals are of different orders. Let us assume the series,
>
> $$..\frac{a}{x^3}, \frac{a}{x^2}, \frac{a}{x}, a, ax, ax^2, ax^3..$$
>
> in which a is a finite constant and x is a variable. If we suppose x to increase, the terms preceding a will diminish, and those following it will increase; when x becomes greater than any assignable quantity, $\frac{a}{x}$ becomes infinitely small *with respect to* a, and because each term bears the same relationship to the one that follows it, every term in the series is infinitely small with respect to the following one, and infinitely great with respect to the preceding one. The quantity ax being *infinitely great* with respect to a finite quantity, is called an infinite of the *first* order; ax^2, ax^3, etc., are infinities of the *second, third*, etc., orders. The quantity $\frac{a}{x}$ being infinitely small with respect to a finite quantity is call an infinitesimal of the *first* order; $\frac{a}{x^2}, \frac{a}{x^3}$, etc., are infinitesimals of the *second*, *third*, etc., orders. (p. 13)

Well, now that we have that straight, we can establish some rules for working with these babies!

If these definitions and explanations have left you reeling, consider the plight of poor students – both his and yours – when we introduce arcane material, like ε and δ, and uniform convergence into our classes. What's more, <u>all</u> authors eventually got to infinitesimals – remember Davies's *dx*. Finally, Cauchy himself defined an infinitesimal as a variable with limit zero,[24] a definition that persisted to 1950 and beyond. While Davies seemed to imagine a real line with holes in it, with Peck, this is not clear and with Cauchy, it's gone. If you are into that kind of thing, you might see if you can bring twentieth century rigor to any of this. It's not easy. I'm not even sure it's possible.

[24] Cauchy, *op. cit.*

What redeems Peck is that he lays out some rules for working with infinitesimals that allow students to manipulate them without having to worry about their ontological status.

"In general the product of an infinitesimal of the m^{th} order by one of the n^{th} order, is an infinitesimal of the $(m+n)^{\text{th}}$ order. The product of a finite quantity by an infinitesimal of the n^{th} order is an infinitesimal of the n^{th} order.

"…Hence, whenever an infinitesimal is connected, by the sign of addition, or subtraction, with a finite quantity, or with an infinitesimal of lower order, it may be suppressed without affecting the value of the expression into which it enters." (p. 14)

We'll see how this works in a minute, but first, let's look at the "General method of Differentiation," which immediately follows this material:

"In order to find the differential of a function, we give to the independent variable its infinitely small increment, and find the corresponding value of the function; from this we subtract the preceding value and reduce the result to its simplest form; we then suppress all infinitesimals which are added to, or subtracted from, those of lower order, and the result is the differential required.

This method of proceeding is too long for general use, and is only employed in deducing rules for differentiation." (pp. 14-15)

Notice that there is no definition of a differential, only rules for finding one.

Here's the quotient rule: We want to find $d\left(\frac{s}{t}\right)$. So, following the instructions,

$\frac{s+ds}{t+dt}-\frac{s}{t}=\frac{st+tds-st-sdt}{t^2+tdt}=\frac{tds-sdt}{t^2+tdt}$. Now we can suppress the *tdt* term, since it is an infinitesimal of the first order, added to a finite quantity. Thus, $d\left(\frac{s}{t}\right)=\frac{tds-sdt}{t^2}$. Now wasn't that easy?

Peck goes on to define the differential coefficient as, of course the quotient $\frac{dy}{dx}$, which makes perfect sense in his system.

One of the advantages infinitesimals had, according to its advocates, was that they made applications easier. These folks envisioned a curve as composed of infinitesimally short lines. The geometry of these lines is exactly the same as the geometry of their big brothers, so using them became only a matter of imitating what we would do as approximations. What is the slope of a tangent line? Why the rise over the run, of course, and that's just exactly what $\frac{dy}{dx}$ is! Similarly, $ds=\sqrt{dx^2+dy^2}$ is not just approximately the differential of arc length, it really is the right thing, by the Pythagorean theorem. And, if you are thinking ahead, you'll see that the advantages for applications of the integral are even greater.

Peck devotes nearly a quarter of his book to applications of the differential calculus, covering the same ground as Davies. Later, at the end of the book, he studies motion, including acceleration and the velocity of a point rolling along a curve.[25]

Integration, of course, is the process by which one might find the function from which a differential may have been derived.[26] There follows 50 pages of techniques of integration. Since differentials of area under a curve, $f(x)dx$, and arc length, $ds = \sqrt{dx^2 + dy^2}$, have been discussed earlier, those applications are immediate, but he's also set up the differentials of volumes of revolution, $dV = \pi y^2 dx,$ and surface area, $dS = 2\pi y ds,$ making those applications easy also.

Perhaps the most distinctive feature of the book is the applications to mechanics and astronomy, the last part of the book. Here, he not only worries about velocity and acceleration, but also center of gravity, moments of inertia, and simple pendulums. Once again, the infinitesimal approach proves valuable, which is the reason you may find physicists and engineers still using it as a way to derive their formulas!

THE MODERN SYNTHESIS

Looking at my list of authors for 1895 to 1910, you'll see some names you might recognize, for example Osgood, a prominent Harvard mathematician and eighth president of the AMS.[27] This is quite a different collection of authors. For example, look at the number of PhDs. Notice also that a number had studied in Europe. This is the period of professionalization in the American mathematics community. The AMS had been formed in 1888, and its membership doubled between 1895 and 1907 to 568.

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[25] see Peck, op. cit., pp.53 – 99 and pp. 171 – 205.
[26] Peck, *ibid.*, p 102.
[27] Achibald, op. cit., p. 153 – 55.