AP® Calculus BC
2005 Scoring Guidelines
Form B

Question 1

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = 12t - 3t^2 \text{ and } \frac{dy}{dt} = \ln\left(1 + (t-4)^4\right).$$

At time $t = 0$, the object is at position $(-13, 5)$. At time $t = 2$, the object is at point P with x-coordinate 3.

(a) Find the acceleration vector at time $t = 2$ and the speed at time $t = 2$.

(b) Find the y-coordinate of P.

(c) Write an equation for the line tangent to the curve at P.

(d) For what value of t, if any, is the object at rest? Explain your reasoning.

(a) $x''(2) = 0,\ y''(2) = -\frac{32}{17} = -1.882$ $a(2) = \langle 0, -1.882 \rangle$ Speed $= \sqrt{12^2 + (\ln(17))^2} = 12.329$ or 12.330	$2 : \begin{cases} 1 : \text{acceleration vector} \\ 1 : \text{speed} \end{cases}$
(b) $y(t) = y(0) + \int_0^t \ln\left(1 + (u-4)^4\right) du$ $y(2) = 5 + \int_0^2 \ln\left(1 + (u-4)^4\right) du = 13.671$	$3 : \begin{cases} 1 : \int_0^2 \ln\left(1 + (u-4)^4\right) du \\ 1 : \text{handles initial condition} \\ 1 : \text{answer} \end{cases}$
(c) At $t = 2$, slope $= \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\ln(17)}{12} = 0.236$ $y - 13.671 = 0.236(x - 3)$	$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{equation} \end{cases}$
(d) $x'(t) = 0$ if $t = 0, 4$ $y'(t) = 0$ if $t = 4$ $t = 4$	$2 : \begin{cases} 1 : \text{reason} \\ 1 : \text{answer} \end{cases}$

Question 2

A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \le t \le 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

(a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?

(b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?

(c) At what time t, for $0 \le t \le 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

(d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

(a) No; the amount of water is not increasing at $t = 15$ since $W(15) - R(15) = -121.09 < 0$.

1 : answer with reason

(b) $1200 + \int_0^{18} (W(t) - R(t))\,dt = 1309.788$

1310 gallons

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $W(t) - R(t) = 0$

$t = 0, 6.4948, 12.9748$

t (hours)	gallons of water
0	1200
6.495	525
12.975	1697
18	1310

The values at the endpoints and the critical points show that the absolute minimum occurs when $t = 6.494$ or 6.495.

$3 : \begin{cases} 1 : \text{interior critical points} \\ 1 : \text{amount of water is least at} \\ \quad t = 6.494 \text{ or } 6.495 \\ 1 : \text{analysis for absolute minimum} \end{cases}$

(d) $\int_{18}^{k} R(t)\,dt = 1310$

$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{equation} \end{cases}$

Question 3

The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n (n-1)^2} \text{ for } n \geq 2.$$

The graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

(a) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

(b) Write the third-degree Taylor polynomial for f about $x = 0$.

(c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

(a) f has a relative maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(b) $f(0) = 6,\ f'(0) = 0$

$$f''(0) = -\frac{3!}{5^2 1^2} = -\frac{6}{25},\ f'''(0) = \frac{4!}{5^3 2^2}$$

$$P(x) = 6 - \frac{3!x^2}{5^2 2!} + \frac{4!x^3}{5^3 2^2 3!} = 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3$$

3 : $P(x)$
$\langle -1 \rangle$ each incorrect term
Note: $\langle -1 \rangle$ max for use of extra terms

(c) $u_n = \dfrac{f^{(n)}(0)}{n!}x^n = \dfrac{(-1)^{n+1}(n+1)}{5^n (n-1)^2}x^n$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\dfrac{(-1)^{n+2}(n+2)}{5^{n+1}n^2}x^{n+1}}{\dfrac{(-1)^{n+1}(n+1)}{5^n(n-1)^2}x^n}\right|$$

$$= \left(\frac{n+2}{n+1}\right)\left(\frac{n-1}{n}\right)^2 \frac{1}{5}|x|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{5}|x| < 1 \text{ if } |x| < 5.$$

The radius of convergence is 5.

$4 : \begin{cases} 1 : \text{general term} \\ 1 : \text{sets up ratio} \\ 1 : \text{computes limit} \\ 1 : \text{applies ratio test to get radius of convergence} \end{cases}$

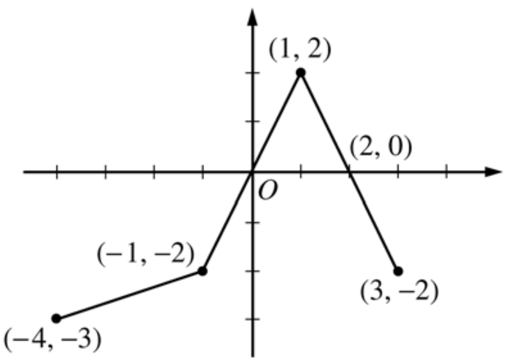
Question 4

The graph of the function f above consists of three line segments.

(a) Let g be the function given by $g(x) = \int_{-4}^{x} f(t)\,dt$.

For each of $g(-1)$, $g'(-1)$, and $g''(-1)$, find the value or state that it does not exist.

(b) For the function g defined in part (a), find the x-coordinate of each point of inflection of the graph of g on the open interval $-4 < x < 3$. Explain your reasoning.



Graph of f

(c) Let h be the function given by $h(x) = \int_{x}^{3} f(t)\,dt$. Find all values of x in the closed interval $-4 \le x \le 3$ for which $h(x) = 0$.

(d) For the function h defined in part (c), find all intervals on which h is decreasing. Explain your reasoning.

(a) $g(-1) = \int_{-4}^{-1} f(t)\,dt = -\frac{1}{2}(3)(5) = -\frac{15}{2}$ $g'(-1) = f(-1) = -2$ $g''(-1)$ does not exist because f is not differentiable at $x = -1$.	$3 : \begin{cases} 1 : g(-1) \\ 1 : g'(-1) \\ 1 : g''(-1) \end{cases}$
(b) $x = 1$ $g' = f$ changes from increasing to decreasing at $x = 1$.	$2 : \begin{cases} 1 : x = 1 \text{ (only)} \\ 1 : \text{reason} \end{cases}$
(c) $x = -1,\ 1,\ 3$	2 : correct values $\langle -1 \rangle$ each missing or extra value
(d) h is decreasing on $[0, 2]$ $h' = -f < 0$ when $f > 0$	$2 : \begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$

Question 5

Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$.

(b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\dfrac{1}{2}$.

(c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

(d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\dfrac{dy}{dt} = 6$. Find the value of $\dfrac{dx}{dt}$ at time $t = 5$.

(a) $2yy' = y + xy'$
$(2y - x)y' = y$
$y' = \dfrac{y}{2y - x}$

$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$

(b) $\dfrac{y}{2y - x} = \dfrac{1}{2}$
$2y = 2y - x$
$x = 0$
$y = \pm\sqrt{2}$
$(0, \sqrt{2}), (0, -\sqrt{2})$

$2 : \begin{cases} 1 : \dfrac{y}{2y - x} = \dfrac{1}{2} \\ 1 : \text{answer} \end{cases}$

(c) $\dfrac{y}{2y - x} = 0$
$y = 0$
The curve has no horizontal tangent since $0^2 \neq 2 + x \cdot 0$ for any x.

$2 : \begin{cases} 1 : y = 0 \\ 1 : \text{explanation} \end{cases}$

(d) When $y = 3$, $3^2 = 2 + 3x$ so $x = \dfrac{7}{3}$.

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \dfrac{y}{2y - x} \cdot \dfrac{dx}{dt}$

At $t = 5$, $6 = \dfrac{3}{6 - \frac{7}{3}} \cdot \dfrac{dx}{dt} = \dfrac{9}{11} \cdot \dfrac{dx}{dt}$

$\left.\dfrac{dx}{dt}\right|_{t=5} = \dfrac{22}{3}$

$3 : \begin{cases} 1 : \text{solves for } x \\ 1 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

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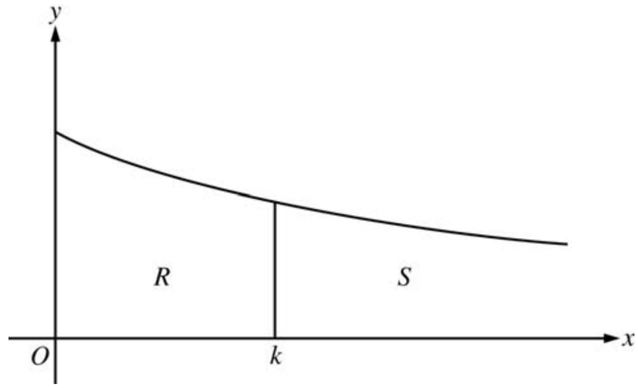
Question 6



Consider the graph of the function f given by $f(x) = \dfrac{1}{x+2}$ for $x \geq 0$, as shown in the figure above. Let R be the region bounded by the graph of f, the x- and y-axes, and the vertical line $x = k$, where $k \geq 0$.

(a) Find the area of R in terms of k.

(b) Find the volume of the solid generated when R is revolved about the x-axis in terms of k.

(c) Let S be the unbounded region in the first quadrant to the right of the vertical line $x = k$ and below the graph of f, as shown in the figure above. Find all values of k such that the volume of the solid generated when S is revolved about the x-axis is equal to the volume of the solid found in part (b).

(a) Area of $R = \displaystyle\int_0^k \frac{1}{x+2}\,dx = \ln(k+2) - \ln(2)$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antidifferentiation and evaluation} \end{cases}$

(b) $$V_R = \pi \int_0^k \frac{1}{(x+2)^2}\,dx$$
$$= -\frac{\pi}{x+2}\Big|_0^k = \frac{\pi}{2} - \frac{\pi}{k+2}$$

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{antidifferentiation and evaluation} \end{cases}$

(c) $$V_S = \pi \int_k^\infty \frac{1}{(x+2)^2}\,dx$$
$$= \lim_{n\to\infty} -\frac{\pi}{x+2}\Big|_k^n = \frac{\pi}{k+2}$$
$$V_S = V_R$$
$$\frac{\pi}{k+2} = \frac{\pi}{2} - \frac{\pi}{k+2}$$
$$\frac{2}{k+2} = \frac{1}{2}$$
$$k = 2$$

$4 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antidifferentiation and evaluation} \\ 1 : \text{equation} \\ 1 : \text{answer} \end{cases}$