## Question 1

An object moving along a curve in the $xy$-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = 12t - 3t^2 \quad \text{and} \quad \frac{dy}{dt} = \ln\left(1 + (t - 4)^4\right).$$

At time $t = 0$, the object is at position $(-13, 5)$. At time $t = 2$, the object is at point $P$ with $x$-coordinate $3$.

(a) Find the acceleration vector at time $t = 2$ and the speed at time $t = 2$.

(b) Find the $y$-coordinate of $P$.

(c) Write an equation for the line tangent to the curve at $P$.

(d) For what value of $t$, if any, is the object at rest? Explain your reasoning.

### Solution

(a) \(x''(2) = 0, \ y''(2) = -\frac{32}{17} = -1.882\)

\[a(2) = \langle 0, -1.882 \rangle\]

Speed \(= \sqrt{12^2 + (\ln(17))^2} = 12.329 \text{ or } 12.330\)

(b) \(y(t) = y(0) + \int_0^t \ln\left(1 + (u - 4)^4\right) \, du\)

\[y(2) = 5 + \int_0^2 \ln\left(1 + (u - 4)^4\right) \, du = 13.671\]

(c) At $t = 2$, slope \(= \frac{dy}{dx} = \frac{\ln(17)}{12} = 0.236\)

\[y - 13.671 = 0.236(x - 3)\]

(d) \(x'(t) = 0 \text{ if } t = 0, 4\)

\[y'(t) = 0 \text{ if } t = 4\]

\[t = 4\]
A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \leq t \leq 18$ hours, water is pumped into the tank at the rate

$W(t) = 95\sqrt{t} \sin^2 \left( \frac{t}{6} \right) \text{ gallons per hour.}$

During the same time interval, water is removed from the tank at the rate

$R(t) = 275\sin^2 \left( \frac{t}{3} \right) \text{ gallons per hour.}$

(a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?

(b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?

(c) At what time $t$, for $0 \leq t \leq 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

(d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let $k$ be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of $k$.

(a) No; the amount of water is not increasing at $t = 15$ since $W(15) - R(15) = -121.09 < 0$.

(b) $1200 + \int_0^{18} (W(t) - R(t)) \, dt = 1309.788$

1310 gallons

(c) $W(t) - R(t) = 0$

$t = 0, 6.4948, 12.9748$

<table>
<thead>
<tr>
<th>$t$ (hours)</th>
<th>gallons of water</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1200</td>
</tr>
<tr>
<td>6.495</td>
<td>525</td>
</tr>
<tr>
<td>12.975</td>
<td>1697</td>
</tr>
<tr>
<td>18</td>
<td>1310</td>
</tr>
</tbody>
</table>

The values at the endpoints and the critical points show that the absolute minimum occurs when $t = 6.494$ or $6.495$.

(d) $\int_{18}^{k} R(t) \, dt = 1310$
The Taylor series about $x = 0$ for a certain function $f$ converges to $f(x)$ for all $x$ in the interval of convergence. The $n$th derivative of $f$ at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1} (n+1)!}{5^n (n-1)^2}$$

for $n \geq 2$.

The graph of $f$ has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

(a) Determine whether $f$ has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

(b) Write the third-degree Taylor polynomial for $f$ about $x = 0$.

(c) Find the radius of convergence of the Taylor series for $f$ about $x = 0$. Show the work that leads to your answer.

(a) $f$ has a relative maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

(b) $f(0) = 6$, $f'(0) = 0$

$$f''(0) = \frac{3!}{5^2 1^2} = \frac{6}{25}$$

$$f'''(0) = \frac{4!}{5^3 2^2}$$

$$P(x) = 6 - \frac{3!x^2}{5^2 2!} + \frac{4!x^3}{5^3 2^2 3!} = 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3$$

(c) $u_n = \frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^{n+1} (n+1)!}{5^n (n-1)^2}x^n$

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(-1)^{n+2} (n+2)!}{5^{n+1} n^2} \frac{5^n (n-1)^2}{(-1)^{n+1} (n+1)!} \right|$$

$$= \left| \frac{n+2}{n+1} \cdot \frac{n-1}{n} \cdot \frac{1}{5} \cdot \frac{1}{x} \right|$$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \frac{1}{5} |x| < 1$$

if $|x| < 5$.

The radius of convergence is 5.
The graph of the function \( f \) above consists of three line segments.

(a) Let \( g \) be the function given by \( g(x) = \int_{-4}^{x} f(t) \, dt \).

For each of \( g(-1), \ g'(-1), \) and \( g''(-1), \) find the value or state that it does not exist.

(b) For the function \( g \) defined in part (a), find the \( x \)-coordinate of each point of inflection of the graph of \( g \) on the open interval \( -4 < x < 3 \). Explain your reasoning.

(c) Let \( h \) be the function given by \( h(x) = \int_{x}^{3} f(t) \, dt \). Find all values of \( x \) in the closed interval \( -4 \leq x \leq 3 \) for which \( h(x) = 0 \).

(d) For the function \( h \) defined in part (c), find all intervals on which \( h \) is decreasing. Explain your reasoning.

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(a) \( g(-1) = \int_{-4}^{-1} f(t) \, dt = -\frac{1}{2}(3)(5) = -\frac{15}{2} \)

\[ g'(-1) = f(-1) = -2 \]

\[ g''(-1) \text{ does not exist because } f \text{ is not differentiable at } x = -1. \]

(b) \( x = 1 \)

\[ g' = f \text{ changes from increasing to decreasing at } x = 1. \]

(c) \( x = -1, 1, 3 \)

2 : correct values

\(-1\) each missing or extra value

(d) \( h \) is decreasing on \([0, 2]\)

\[ h' = -f < 0 \text{ when } f > 0 \]

2 : interval

1 : reason
Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\frac{dy}{dx} = \frac{y}{2y - x}$.

(b) Find all points $(x, y)$ on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

(c) Show that there are no points $(x, y)$ on the curve where the line tangent to the curve is horizontal.

(d) Let $x$ and $y$ be functions of time $t$ that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of $y$ is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time $t = 5$.

(a) $2y y' = y + xy'$
\[(2y - x) y' = y\]
\[y' = \frac{y}{2y - x}\]

(b) $\frac{y}{2y - x} = \frac{1}{2}$
\[2y = 2y - x\]
\[x = 0\]
\[y = \pm \sqrt{2}\]
\[(0, \sqrt{2}), (0, -\sqrt{2})\]

(c) $\frac{y}{2y - x} = 0$
\[y = 0\]
The curve has no horizontal tangent since $0^2 \neq 2 + x \cdot 0$ for any $x$.

(d) When $y = 3$, $3^2 = 2 + 3x$ so $x = \frac{7}{3}$.
\[\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{y}{2y - x} \cdot \frac{dx}{dt}\]
At $t = 5$, $6 = \frac{3}{6 - \frac{7}{3}} \cdot \frac{dx}{dt} = \frac{9}{11} \cdot \frac{dx}{dt}$
\[\frac{dx}{dt} \bigg|_{t=5} = \frac{22}{3}\]
Consider the graph of the function \( f \) given by
\[ f(x) = \frac{1}{x + 2} \quad \text{for } x \geq 0, \]
as shown in the figure above. Let \( R \) be the region bounded by the graph of \( f \), the \( x \)- and \( y \)-axes, and the vertical line \( x = k \), where \( k \geq 0 \).

(a) Find the area of \( R \) in terms of \( k \).

(b) Find the volume of the solid generated when \( R \) is revolved about the \( x \)-axis in terms of \( k \).

(c) Let \( S \) be the unbounded region in the first quadrant to the right of the vertical line \( x = k \) and below the graph of \( f \), as shown in the figure above. Find all values of \( k \) such that the volume of the solid generated when \( S \) is revolved about the \( x \)-axis is equal to the volume of the solid found in part (b).

(a) Area of \( R \)
\[
\int_{0}^{k} \frac{1}{x + 2} \, dx = \ln (k + 2) - \ln (2)
\]

(b) Volume of the solid
\[
V_R = \pi \int_{0}^{k} \frac{1}{(x + 2)^2} \, dx
\]
\[
= -\frac{\pi}{x + 2} \bigg|_{0}^{k} = \frac{\pi}{2} - \frac{\pi}{k + 2}
\]

(c) Volume of the solid
\[
V_S = \pi \int_{k}^{\infty} \frac{1}{(x + 2)^2} \, dx
\]
\[
= \lim_{n \to \infty} -\frac{\pi}{x + 2} \bigg|_{k}^{n} = \frac{\pi}{k + 2}
\]
\[
V_S = V_R
\]
\[
\frac{\pi}{k + 2} = \frac{\pi}{2} - \frac{\pi}{k + 2}
\]
\[
\frac{2}{k + 2} = \frac{1}{2}
\]
\[
k = 2
\]